Fusion Rules for the Charge Conjugation Orbifold

Toshiyuki Abe

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
e-mail: sm3002at@ecs.cmc.osaka-u.ac.jp

Abstract

We completely determine fusion rules for irreducible modules of the charge conjugation orbifold.

1 Introduction

The charge conjugation orbifold V_L^+ is the orbifold of the lattice vertex operator algebra V_L associated to a rank one even lattice L by the automorphism θ given by extending the -1-isometry of L (cf. [KT, Section 6.1]). The set of all equivalence class of irreducible V_L^+-modules consists of $k+3$ modules derived from irreducible (untwisted) V_L-modules (we call them untwisted type modules) and 4 modules from irreducible θ-twisted modules (we call them twisted type modules) (see [DN2]), where k is the half square length of the generator of L. In this paper we completely determine the fusion rule for the irreducible V_L^+-modules. The intertwining operators for V_L-modules constructed in [DL] give rise to intertwining operators for untwisted type modules. We construct intertwining operators involving twisted type modules by means of the twisted intertwining operators constructed in [FLM]. The fusion rules and explicit forms of intertwining operators for the free bosonic orbifold vertex operator algebra $M(1)^+$ determined in [A] play important roles in analyzing intertwining operators for V_L^+.

The vertex operator algebra V_L^+ and its irreducible modules are constructed as follows: Let $L=\mathbb{Z}\alpha$ be a rank one even lattice with a $\mathbb{Z}$-bilinear form $\langle\cdot\,,\cdot\rangle$ defined by $\langle\alpha,\alpha\rangle=2k$ for a positive integer k. Set $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}L$ and extend the $\mathbb{Z}$-bilinear form to a $\mathbb{C}$-bilinear form on $\mathfrak{h}$ in the canonical way. Let $\hat{\mathfrak{h}}=\mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ be its affinization with the center K.

Then the Fock space $M(1) = S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$ is a simple vertex operator algebra with central charge 1. Let $\mathbb{C}[\mathfrak{h}] = \oplus_{\lambda\in\mathfrak{h}}\mathbb{C}e_\lambda$ be the group algebra of the abelian group $\mathfrak{h}$, and set $\mathbb{C}[M] = \oplus_{\lambda\in M}\mathbb{C}e_\lambda$ for a subset M of $\mathfrak{h}$. It is known that $V_L = M(1)\otimes\mathbb{C}[L]$ is a simple vertex operator algebra with central charge 1 (cf. [FLM]), and $V_{\lambda+L} = M(1)\otimes\mathbb{C}[\lambda+L]$ is an irreducible V_L-module for all $\lambda\in L^\circ$, where L° is the dual lattice of L. Moreover all irreducible V_L-modules are given by the set $\{\, V_{\lambda+L} \mid \lambda + L \in L^\circ/L \,\}$ (cf. [D1]). Let θ be the involution of L defined by $\theta(\beta) = -\beta$ for $\beta\in L$. Then the involution θ can be lifted to an isomorphism of V_{L°, and the θ-invariant subspace of V_L becomes a simple vertex operator algebra with central charge 1, denoted by V_L^+. The automorphism θ induces an V_L^+-module isomorphism from $V_{\beta+L}$ to $V_{-\beta+L}$ for $\beta\in L^\circ$. For a θ-invariant subspace W of $V_{L^{circ}}$, we denote the ±1-eigenspaces by $W^\pm$ respectively. Then $V_L^\pm$, $V_{\alpha/2+L}^\pm$ and $V_{r\alpha/2k+L}$ for $1\le r\le k-1$ are irreducible V_L^+-modules (see [DN2]).

Let $\hat{\mathfrak{h}}[-1] = \mathfrak{h}\otimes t^{1/2}\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ be the twisted affine Lie algebra and set $M(1)(\theta) = S(\mathfrak{h}\otimes t^{-1/2}\mathbb{C}[t^{-1}])$. Then $M(1)(\theta)$ is a unique irreducible θ-twisted $M(1)$-module (cf. [FLM] and [D2]). The automorphism θ acts on $M(1)(\theta)$, and the ±1-eigenspaces $M(1)(\theta)^\pm$ become irreducible $M(1)^+$-modules (see [DN1]). Let T^1 and T^2 be irreducible $\mathbb{C}[L]$-modules on which e_α acts 1 and -1 respectively. Then the tensor products $V_L^{T^i} = M(1)(\theta)\otimes T^i$ ($i = 1,2$) are irreducible θ-twisted V_L-modules, and their ±1-eigenspaces $V_L^{T^i,\pm}$ for θ become irreducible V_L^+-modules ([DN2]). In [DN2], it is proved that every irreducible V_L^+-module is isomorphic to one of the irreducible modules $V_L^\pm, V_{\alpha/2+L}^\pm, V_{r\alpha/2k+L}$ for $1\le r\le k-1$ and $V_L^{T_i,\pm}$ for $i=1,2$.

For a vertex operator algebra V and its modules W^1, W^2 and W^3, the dimension of the vector space $I_V\binom{W^3}{W^1\ W^2}$ of all intertwining operators of type $\binom{M^3}{M^1\ M^2}$ is called the fusion rule of corresponding type and denoted by $N_{W^1W^2}^{W^3}$. It is known that fusion rules have the following symmetry;

$$I_V\binom{W^3}{W^1\ W^2} \cong \binom{W^3}{W^2\ W^1} \cong \binom{(W^2)}{W^1\ (W^3)'}, \tag{1.1}$$

where W' means the contragredient module of W (see [FHL, HL]).

We give the correspondence of irreducible V_L^+-modules and contragredient modules (see Proposition 2.8). The correspondence and the symmetry of fusion rules (1.1) are very useful in reducing the arguments to determine the fusion rules for V_L^+.

We explain the method of determining the fusion rules for V_L^+ in more detail. Let

W^1, W^2 and W^3 be V_L^+-modules, and suppose that W^1 and W^2 contain $M(1)^+$-submodules M^1 and M^2 respectively. Then we have a canonical restriction map

$$I_{V_L^+}\binom{W^3}{W^1\ \ W^2} \to I_{M(1)^+}\binom{W^3}{M^1\ \ M^2},\ \mathcal{Y} \mapsto \mathcal{Y}|_{N^1\otimes N^2}.$$

It is known that if W^1 and W^2 are irreducible, the restriction map is injective (cf. [DL], Proposition 11.9). Therefore we then have

$$\dim I_{V_L^+}\binom{W^3}{W^1\ \ W^2} \leq \dim I_{M(1)^+}\binom{W^3}{M^1\ \ M^2}. \tag{1.2}$$

We also prove that all irreducible V_L^+-modules are completely reducible as $M(1)^+$-modules and that the multiplicity of each irreducible $M(1)^+$-module is at most one. Using this fact, (1.2) and fusion rules for $M(1)^+$, we show that fusion rules for V_L^+ are zero or one. The formula (1.2) also shows that for irreducible V_L^+-modules W^1, W^2 and W^3, if there are $M(1)^+$-submodules M^1 of W^1 and M^2 of W^2 such that the fusion rule $N_{M^1M^2}^{W^3}$ for $M(1)^+$ is zero, then the fusion rule $N_{W^1W^2}^{W^3}$ for V_L^+ is zero. For almost of irreducible V_L^+-modules W^1, W^2 and W^3 for which the fusion rule $N_{W^1W^2}^{W^3}$ is zero, we can find such $M(1)^+$-submodules M^1 of W^1 and M^2 of W^2. But there are irreducible modules W^1, W^2 and W^3 such that the fusion rule $N_{W^1W^2}^{W^3}$ is zero although the fusion rule $N_{M^1M^2}^{W^3}$ is nonzero for any $M(1)^+$-submodules M^1 of W^1 and M^2 of W^2 (for example $W^1 = V_L^-$, $W^2 = W^3 = V_{\alpha/2+L}^+$). In such case, to show that intertwining operators $\mathcal{Y}$ of corresponding type are zero, we choose irreducible $M(1)^+$-submodules M^1 and M^2 from W^1 and W^2 respectively. Since the fusion rule $N_{M^1M^2}^{W^3}$ is nonzero and W^3 is a direct sum of irreducible $M(1)^+$-modules as $W^3 = \oplus_{i\in I} M_i^3$, we see that the restriction of $\mathcal{Y}$ to $M^1 \otimes M^2$ is a linear combination of intertwining operators of types $\binom{M_i^3}{M^1\ \ M^2}$. Then the explicit forms of intertwining operators of types $\binom{M_i^3}{M^1\ \ M^2}$ shows $\mathcal{Y} = 0$.

The nonzero fusion rules is provided by constructing nontrivial intertwining operators explicitly. The constructions is separated in two cases; one is the case all modules are of untwisted types, and other is the case some modules are of twisted types.

In the case all modules are of untwisted types, the nontrivial intertwining operators are essentially given in [DL]. [DL] construct a nontrivial intertwining operator $\mathcal{Y}_{\lambda\mu}$ for V_L of type $\binom{V_{\lambda+\mu+L}}{V_{\lambda+L}\ \ V_{\mu+L}}$ for $\lambda, \mu \in L^\circ$. The intertwining operator $\mathcal{Y}_{\lambda\mu}$ gives rise to a nonzero intertwining operator for V_L^+ of type $\binom{V_L^{\lambda+\mu}}{V_{\lambda+L}\ \ V_{\mu+L}}$. Since θ induces a V_L^+-module isomorphism from $V_{\lambda+L}$ to $V_{-\lambda+L}$ for $\lambda \in L^\circ$, the operator $\mathcal{Y}_{\lambda,-\mu} \circ \theta$ defined by

$\mathcal{Y}_{\lambda,-\mu} \circ \theta(u,z)v = \mathcal{Y}_{\lambda,-\mu}(u,z)\theta(v)$ for $u \in V_{\lambda+L}$ and $v \in V_{\mu+L}$ gives a nonzero intertwining operator of type $\left({V_L^{\lambda-\mu} \atop V_{\lambda+L} \ \ V_{\mu+L}} \right)$. Then all nonzero intertwining for untwisted type modules are given by restricting $\mathcal{Y}_{\lambda\mu}$ or $\mathcal{Y}_{\lambda,-\mu} \circ \theta$ to irreducible V_L^+-modules.

In the case some modules are of twisted types, we construct nonzero intertwining operators as follows: In [A], an intertwining operator $\mathcal{Y}^\theta$ for $M(1)^+$ of type $\left({M(1)(\theta) \atop M(1,\lambda) \ \ M(1)(\theta)} \right)$ for $\lambda \in L^\circ$ is constructed following [FLM]. As in [DL], for $\lambda \in L^\circ$, we give an linear isomorphism ψ_λ of $T^1 \oplus T^2$ which satisfies $e_\alpha \psi_\lambda = (-1)^{\langle \alpha, \lambda \rangle} \psi_\lambda e_\alpha = \psi_{\lambda+\alpha}$, and define $\tilde{\mathcal{Y}}$ by $\tilde{\mathcal{Y}}(u,z) = \mathcal{Y}^\theta(u,z) \otimes \psi_\gamma$ for $\gamma \in \lambda + L$ and $u \in M(1,\gamma)$. Then for $\lambda \in L^\circ$ and $i,j = 1,2$ which satisfy $(-1)^{\langle \lambda, \alpha \rangle + \delta_{i,j}+1} = 1$, $\tilde{\mathcal{Y}}$ gives rise to an intertwining operator of type $\left({V_L^{T_j} \atop V_{\lambda+L} \ \ V_L^{T_i}} \right)$, and all nonzero intertwining operators in this case are given by restricting $\tilde{\mathcal{Y}}$ to irreducible V_L^+-modules and by using symmetry of fusion rules (1.1).

The organization of this paper is as follows: We recall definitions of modules for a vertex operator algebra and fusion rules in Section 2.1, we review the vertex operator algebras $M(1)^+$ and V_L^+ and their irreducible modules in Section 2.2. In Section 2.3 we state the fusion rules for $M(1)^+$, and discuss the contragredient modules for V_L^+. In Section 3.1, we give the irreducible decompositions of irreducible V_L^+-modules as $M(1)^+$-modules and prove that the fusion rules for V_L^+ are zero or one. In Section 3.2, we state the main theorem (Theorem 3.4). In Section 3.3 and 3.4, the proof of the main theorem is given. In Section 3.3, we determine the fusion rule $N_{W^1 W^2}^{W^3}$ for untwisted type modules W^i $(i = 1,2,3)$. In Section 3.4, fusion rule of type $\left({W^3 \atop W^1 \ \ W^2} \right)$ are determine in the case some of W^i $(i = 1,2,3)$ are twisted type module.

2 Preliminaries

In Section 2.1, we recall the definition of a g-twisted module for a vertex operator algebra and its automorphism g of finite order and that of an intertwining operator following [FLM, FHL, DMZ] and [DLM]. In Section 2.2, we review constructions of vertex operator algebras $M(1)^+$, V_L^+ and their irreducible modules following [FLM, DL, DN1, DN2]. In Section 2.3, we state the fusion rules for $M(1)^+$ obtained in [A] (see Theorem 2.7) and discuss the contragredient modules for V_L^+.

Throughout this paper, $\mathbb{N}$ is the set of nonnegative integers and $\mathbb{Z}_+$ is the set of positive integers.

2.1 Modules, intertwining operators and fusion rules

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and g an automorphism of V of order T. Then V is decomposed into the direct sum of eigenspaces for g:

$$V = \bigoplus_{r=0}^{T-1} V^r,\ V^r = \{\ a \in V \mid g(a) = e^{-\frac{2\pi i r}{T}} a\ \}.$$

A *g-twisted V-module* is a $\mathbb{C}$-graded vector space $M = \oplus_{\lambda \in \mathbb{C}} M(\lambda)$ such that each $M(\lambda)$ is finite dimensional and for fixed $\lambda \in \mathbb{C}$, $M(\lambda + n/T) = 0$ for sufficiently small integer n, and equipped with a linear map

$$\begin{aligned} Y_M : V &\to (\mathrm{End}\, M)\{z\}, \\ a &\mapsto Y_M(a, z) = \sum_{n \in \mathbb{Q}} a_n^M z^{-n-1},\ (a_n^M \in \mathrm{End}\, M) \end{aligned}$$

such that the following conditions hold for $0 \le r \le T-1,\ a \in V^r, b \in V$ *and* $u \in M$:

$$Y_M(a, z) = \sum_{n \in r/T + \mathbb{Z}} a_n^M z^{-n-1},\ Y_M(a, z)v \in z^{-\frac{r}{T}} M((z)),$$

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1) Y_M(b, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(b, z_2) Y_M(a, z_1) \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^{-\frac{r}{T}} Y_M(Y(a, z_0)b, z_2), \end{aligned}$$

$$Y_M(\mathbf{1}, z) = \mathrm{id}_M,$$

$$L(0)v = \lambda v \text{ for } v \in M(\lambda),$$

where we set $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$.

A g-twisted V-module is denoted by (M, Y_M) or simply by M. In the case g is the identity of V, a g-twisted V-module is called a *V-module*. An element $u \in M(\lambda)$ is called a *homogeneous element of weight* λ. We denote the weight by $\lambda = \mathrm{wt}(u)$. We write the component operator a_n^M $(a \in V,\ n \in \mathbb{Q})$ by a_n for simplicity.

For a V-module M, it is known that the restricted dual $M' = \oplus_{\lambda \in \mathbb{C}} M(\lambda)^*$ with the vertex operator $Y_M^*(a, z)$ for $a \in V$ defined by

$$\langle Y_M^*(a, z)u', v\rangle = \langle u', Y_M(e^{zL(1)}(-z^{-2})^{L(0)} a, z^{-1})v\rangle$$

for $u' \in M'$, $v \in M$ is a V-module (cf. [FHL]). The V-module (M', Y_M^*) is called the *contragredient module of* M. The double contragredient module $(M')'$ of M is naturally isomorphic to M, and therefore if M is irreducible, then M' is also irreducible (see [FHL]).

Definition 2.1. *Let* V *be a vertex operator algebra and* (M^i, Y_{M^i}) $(i = 1, 2, 3)$ *be* V*-modules. An* intertwining operator for V of type $\left({M^3 \atop M^1 \ M^2} \right)$ *is a linear map* $\mathcal{Y} : M^1 \otimes M^2 \to M^3\{z\}$*, or equivalently,*

$$\begin{aligned} \mathcal{Y} : M^1 &\to (\mathrm{Hom}\,(M^2, M^3))\{z\}, \\ v &\mapsto \mathcal{Y}(v, z) = \sum_{n \in \mathbb{C}} v_n z^n \ (v_n \in \mathrm{Hom}\,(M^2, M^3)) \end{aligned}$$

such that for $a \in V, v \in M^1$ *and* $u \in M^2$*, following conditions are satisfied:*

For fixed $n \in \mathbb{C}$, $v_{n+k}u = 0$ *for sufficiently large integer* k*,*

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M^3}(a, z_1)\mathcal{Y}(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) \mathcal{Y}(v, z_2)Y_{M^2}(a, z_1) \\ & = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \mathcal{Y}(Y_{M^1}(a, z_0)v, z_2), \end{aligned} \tag{2.1}$$

$$\frac{d}{dz}\mathcal{Y}(v, z) = \mathcal{Y}(L(-1)v, z).$$

The vector space of all intertwining operators of type $\left({M^3 \atop M^1 \ M^2} \right)$ is denoted by $I_V\left({M^3 \atop M^1 \ M^2} \right)$. The dimension of the vector space $I_V\left({M^3 \atop M^1 \ M^2} \right)$ is called the *fusion rule* of corresponding type and denoted by $N_{M^1 M^2}^{M^3}$. Fusion rules have the following symmetry (see [FHL] and [HL]).

Proposition 2.2. *Let* M^i $(i = 1, 2, 3)$ *be* V*-modules. Then there exist natural isomorphisms*

$$I_V\left({M^3 \atop M^1 \ M^2} \right) \cong I_V\left({M^3 \atop M^2 \ M^1} \right) \text{ and } I_V\left({M^3 \atop M^1 \ M^2} \right) \cong I_V\left({(M^2)' \atop M^1 \ (M^3)'} \right).$$

The following lemma is often used in later sections.

Lemma 2.3. ([DL]) *Let* V *be a vertex operator algebra, and let* M^1 *and* M^2 *be irreducible* V*-modules and* M^3 *a* V*-module. If* $\mathcal{Y}$ *is a nonzero intertwining operator of type* $\left({M^3 \atop M^1 \ M^2} \right)$*, then* $\mathcal{Y}(u, z)v \neq 0$ *for any nonzero vectors* $u \in M^1$ *and* $v \in M^2$*.*

As a direct consequence of Lemma 2.3, we have

Corollary 2.4. *Let V, M^i $(i = 1, 2, 3)$ be as in Lemma 2.3, and let U be a vertex operator subalgebra of V with same Virasoro element, N^i a U-submodule of M^i for $i = 1, 2$. Then the restriction map*

$$I_V\binom{M^3}{M^1\ M^2} \to I_U\binom{M^3}{N^1\ N^2},\ \mathcal{Y} \mapsto \mathcal{Y}|_{N^1\otimes N^2},$$

is injective. In particular, we have

$$\dim I_V\binom{M^3}{M^1\ M^2} \le \dim I_U\binom{M^3}{N^1\ N^2}.$$

Let V, M^i $(i = 1, 2, 3)$, U and N^i $(i = 1, 2)$ be as in Corollary 2.4. Suppose that M^3 is decomposed into a direct sum of irreducible U-modules as $M^3 = \oplus_i L^i$. Then there is an isomorphism

$$I_U\binom{\oplus_i L^i}{N^1\ N^2} \cong \oplus_i I_U\binom{L^i}{N^1\ N^2}.$$

Therefore by Corollary 2.4, we have an in equality

$$\dim I_V\binom{M^3}{M^1\ M^2} \le \sum_i \dim I_U\binom{L^i}{N^1\ N^2}. \tag{2.2}$$

Another consequence of Lemma 2.3 is

Lemma 2.5. *Let V be a simple vertex operator algebra, and let M^1 and M^2 be irreducible V-modules. If the fusion rule of type $\binom{M^2}{V\ M^1}$ is nonzero, then M^1 and M^2 are isomorphic to each other as V-modules.*

Proof. Let $\mathcal{Y}$ be an intertwining operator of type $\binom{M^2}{V\ M^1}$. Consider the operator $\mathcal{Y}(\mathbf{1}, z)$. By the $L(-1)$-derivative property (2.2), we see that $\mathcal{Y}(\mathbf{1}, z)$ is independent on z. Denote $f = \mathcal{Y}(\mathbf{1}, z) \in \mathrm{Hom}\,(M^1, M^2)$. Since V is simple and M^1 is irreducible, Proposition 2.3 implies that f is nonzero. By Jacobi identity (2.1), we have a commutation relation

$$[a_n, \mathcal{Y}(\mathbf{1}, z)] = \sum_{i=0}^{\infty} \binom{n}{i} \mathcal{Y}(a_i\mathbf{1}, z) z^{n-i} = 0$$

for $a \in V$ and $n \in \mathbb{Z}$. Hence f is a nonzero V-module homomorphism from M^1 to M^2. Since M^1 and M^2 are irreducible, f is in fact isomorphism. Therefore M^1 is isomorphic to M^2. $\square$

2.2 Vertex operator algebra V_L^+ and its irreducible modules

We discuss the constructions of vertex operator algebras $M(1)$, V_L and their irreducible (twisted) modules following [FLM, DL, D1] and [D2]. We also refer to the vertex operator algebras $M(1)^+$, V_L^+ and irreducible modules for them classified in [DN1, DN2].

Let L be an even lattice of rank 1 with a nondegenerate positive definite $\mathbb{Z}$-bilinear form $\langle\cdot\,,\cdot\rangle$, and $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$. Then $\mathfrak{h}$ has the nondegenerate symmetric $\mathbb{C}$-bilinear form given by extending the form $\langle\cdot\,,\cdot\rangle$ of L. Let $\mathbb{C}[\mathfrak{h}]$ be the group algebra of $\mathfrak{h}$ with a basis $\{\, e_\lambda \,|\, \lambda\in\mathfrak{h}\,\}$. For a subset M of $\mathfrak{h}$, set $\mathbb{C}[M]=\oplus_{\lambda\in M}\mathbb{C}e_\lambda$.

Let $\hat{\mathfrak{h}} = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ be a Lie algebra with the commutation relation given by $[X\otimes t^m, X'\otimes t^n] = m\,\delta_{m+n,0}\,\langle X,X'\rangle\,K, [K,\hat{\mathfrak{h}}]=0$ for $X,X'\in\mathfrak{h}$ and $m,n\in\mathbb{Z}$. Then $\hat{\mathfrak{h}}^+ = \mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}K$ is a subalgebra of $\hat{\mathfrak{h}}$, and the group algebra $\mathbb{C}[\mathfrak{h}]$ becomes a $\hat{\mathfrak{h}}^+$-module by the action $\rho(X\otimes t^n)e_\lambda = \delta_{n,0}\langle X,\lambda\rangle e_\lambda$ and $\rho(K)e_\lambda = e_\lambda$ for $\lambda\in\mathfrak{h}$, $X\in\mathfrak{h}$ and $n\in\mathbb{N}$. It is clear that for a subset M of $\mathfrak{h}$ the subspace $\mathbb{C}[M]$ is a $\hat{\mathfrak{h}}^+$-submodule of $\mathbb{C}[\mathfrak{h}]$. Set V_M the induced module of $\hat{\mathfrak{h}}$ by $\mathbb{C}[M]$:

$$V_M = U(\hat{\mathfrak{h}})\otimes_{U(\hat{\mathfrak{h}}^+)}\mathbb{C}[M]\cong S(\mathfrak{h}\otimes t^{-1}\mathbb{C}[t^{-1}])\otimes\mathbb{C}[M]\ \text{(linearly)},$$

where $U(\cdot\,)$ means a universal enveloping algebra. Denote the action of $X\otimes t^n$ ($X\in\mathfrak{h}, n\in\mathbb{Z}$) on $V_{\mathfrak{h}}$ by $X(n)$ and set $X(z)=\sum_{n\in\mathbb{Z}}X(n)z^{-n-1}$ for $X\in\mathfrak{h}$. For $\lambda\in\mathfrak{h}$, the vertex operator associated with e_λ is defined by

$$\mathcal{Y}^\circ(e_\lambda,z)=\exp\left(\sum_{n=1}^{\infty}\frac{\lambda(-n)}{n}z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}z^{-n}\right)e_\lambda z^{\lambda(0)}, \tag{2.3}$$

where e_λ in the right-hand side means the left multiplication of $e_\lambda\in\mathbb{C}[\mathfrak{h}]$ on the group algebra $\mathbb{C}[\mathfrak{h}]$, and $z^{\lambda(0)}$ is an operator on $V_{\mathfrak{h}}$ defined by $z^{\lambda(0)}u = z^{\langle\lambda,\mu\rangle}u$ for $\mu\in\mathfrak{h}$ and $u\in M(1,\mu)$. For $v = X_1(-n_1)\cdots X_m(-n_m)\,e_\lambda\in V_{\mathfrak{h}}$ ($X_i\in\mathfrak{h}$ and $n_i\in\mathbb{Z}_+$), the corresponding vertex operator is defined by

$$\mathcal{Y}^\circ(v,z) = {}^\circ_\circ\,\partial^{(n_1-1)}X_1(z)\cdots\partial^{(n_m-1)}X_m(z)\mathcal{Y}^\circ(e_\lambda,z)\,{}^\circ_\circ, \tag{2.4}$$

where $\partial^{(n)}=(\frac{1}{n!})(d/dz)^n$, and the normal ordering ${}^\circ_\circ\cdot{}^\circ_\circ$ is an operation which reorders so that $X(n)$ ($X\in\mathfrak{h}, n<0$) and e_λ to be placed to the left of $X(n)$ ($X\in\mathfrak{h}, n\geq 0$) and $z^{\lambda(0)}$. We extend $\mathcal{Y}^\circ$ to $V_{\mathfrak{h}}$ by linearity. We denote $Y(a,z)=\mathcal{Y}^\circ(a,z)$ when a is in V_L.

Set $L = \mathbb{Z}\alpha$ and $\langle\alpha, \alpha\rangle = 2k$ for $k \in \mathbb{Z}_+$, and $L^\circ = \{\lambda \in \mathfrak{h} \mid \langle\lambda, \alpha\rangle \in \mathbb{Z}\}$, the dual lattice of L. Let $h = \alpha/\sqrt{2k}$ be the orthonormal basis of $\mathfrak{h}$ and set $\mathbf{1} = 1 \otimes e_0$ and $\omega = (1/2)\, h(-1)^2 e_0$. Then $(V_L, Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra with central charge 1 and for $\lambda + L \in L^\circ/L$, $(V_{\lambda+L}, Y)$ is an irreducible module for V_L. Furthermore $V_{\lambda+L}$ for $\lambda + L \in L^\circ/L$ give all inequivalent irreducible V_L-modules. Set $M(1) = S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes e_0 \subset V_L$, then $(M(1), Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra. If we set $M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^+)} \mathbb{C}e_\lambda$ for each $\lambda \in \mathfrak{h}$, then $(M(1, \lambda), \mathcal{Y}^\circ)$ becomes an irreducible $M(1)$-module (see [D1, DL]).

Let θ be a linear isomorphism of $V_\mathfrak{h}$ defined by

$$\theta(X_1(-n_1)X_2(-n_2)\cdots X_\ell(-n_\ell) \otimes e_\lambda) = (-1)^\ell\, X_1(-n_1)X_2(-n_2)\cdots X_\ell(-n_\ell) \otimes e_{-\lambda},$$

for $X_i \in \mathfrak{h}, n \in \mathbb{Z}_+$ and $\lambda \in \mathfrak{h}$. Then θ induces automorphisms of V_L and $M(1)$. For a θ-invariant subspace W of $V_\mathfrak{h}$, we denote the ± 1-eigenspaces of W for θ by $W^\pm$. Then $(V_L^+, Y, \mathbf{1}, \omega)$ and $(M(1)^+, Y, \mathbf{1}, \omega)$ are vertex operator algebras. Furthermore $M(1)^\pm$ and $M(1, \lambda)$ for $\lambda \neq 0$ are irreducible $M(1)^+$-modules, and θ induces an $M(1)^+$-module isomorphism between $M(1, \lambda)$and $M(1, -\lambda)$ (see[DN1]). As to V_L^+-modules, $V_L^\pm$, $V_{\alpha/2+L}^\pm$ and $V_{r\alpha/2k+L}$ for $1 \le r \le k-1$ are irreducible modules (see [DN2]) and θ induces a V_L^+-module isomorphism between $V_{\lambda+L}$ and $V_{-\lambda+L}$ for $\lambda \in L^\circ$.

Now we review the construction of θ-twisted V_L-modules following [FLM, D2]. Let $\hat{\mathfrak{h}}[-1] = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be a Lie algebra with commutation relation $[X \otimes t^m, X' \otimes t^n] = m\, \delta_{m+n,0}\, \langle X, X'\rangle\, K$, $[K, \hat{\mathfrak{h}}[-1]] = 0$ for $X, X' \in \mathfrak{h}$ and $m, n \in 1/2 + \mathbb{Z}$. Then $\mathbb{C}$ becomes a one-dimensional module for $\hat{\mathfrak{h}}[-1]^+ = \mathfrak{h} \otimes t^{1/2}\,\mathbb{C}[t] \oplus \mathbb{C}K$ by defining the actions by $\rho(X \otimes t^n)1 = 0$ and $\rho(K)\,1 = 1$ for $X \in \mathfrak{h}$ and $n \in 1/2 + \mathbb{N}$. Set $M(1)(\theta)$ the induced $\hat{\mathfrak{h}}[-1]$-module :

$$M(1)(\theta) = U(\hat{\mathfrak{h}}[-1]) \otimes_{U(\hat{\mathfrak{h}}[-1]^+)} \mathbb{C} \cong S\left(\mathfrak{h} \otimes t^{-\frac{1}{2}}\mathbb{C}[t^{-1}]\right) \quad \text{(linearly)}.$$

Denote the action of $X \otimes t^n$ $(X \in \mathfrak{h}, n \in 1/2 + \mathbb{Z})$ on $M(1)(\theta)$ by $X(n)$, and set $X(z) = \sum_{n\in 1/2+\mathbb{Z}} X(n)\, z^{-n-1}$. For $\lambda \in L^\circ$ a twisted vertex operator associated with $e_\lambda \in V_\mathfrak{h}$ is defined by

$$\mathcal{Y}^\theta(e_\lambda, z) = 2^{-\langle\lambda,\lambda\rangle} z^{-\frac{\langle\lambda,\lambda\rangle}{2}} \exp\left(\sum_{n\in 1/2+\mathbb{N}} \frac{\lambda(-n)}{n} z^n\right) \exp\left(-\sum_{n\in 1/2+\mathbb{N}} \frac{\lambda(n)}{n} z^{-n}\right). \qquad (2.5)$$

For $v = X_1(-n_1)\cdots X_m(-n_m)e_\lambda \in V_{L^\circ}$ ($X_i \in \mathfrak{h}$ and $n_i \in \mathbb{Z}_+$), set

$$W^\theta(v,z) = {\circ \atop \circ}\partial^{(n_1-1)}X_1(z)\cdots\partial^{(n_m-1)}X_m(z)\mathcal{Y}^\theta(v_\lambda,z){\circ \atop \circ}, \tag{2.6}$$

and extend it to V_{L° by linearity, where the normal ordering ${\circ \atop \circ}\cdot{\circ \atop \circ}$ is an operation which reorders so that $X(n)$ ($X \in \mathfrak{h}, n < 0$) to be placed to the left of $X(n)$ ($X \in \mathfrak{h}, n > 0$). Let $c_{mn} \in \mathbb{Q}$ be coefficients defined by the formal power series expansion

$$\sum_{m,n\geq 0} c_{mn}x^my^n = -\log\left(\frac{(1+x)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}}{2}\right),$$

and set $\Delta_z = \sum_{m,n\geq 0} c_{mn}h(m)h(n)z^{-m-n}$. Then the twisted vertex operator associated to $u \in V_{L^\circ}$ is defined by

$$\mathcal{Y}^\theta(u,z) = W^\theta(\exp(\Delta_z)u,z). \tag{2.7}$$

If we write $Y^\theta(a,z) = \mathcal{Y}^\theta(a,z)$ for $a \in M(1)$, the pair $(M(1)(\theta), Y^\theta)$ is the unique irreducible θ-twisted $M(1)$-module.

Let T_1 and T_2 be irreducible $\mathbb{C}[L]$-modules which e_α acts as 1 and -1 respectively, and set $V_L^{T_i} = M(1)(\theta)\otimes_{\mathbb{C}} T_i$ for $i = 1,2$. For $u \in M(1,\beta)$ ($\beta \in L$), the corresponding twisted vertex operator is defined by $Y^\theta(u,z) = \mathcal{Y}^\theta(u,z)\otimes e_\beta$. We extend Y^θ to V_L by linearity. Then $(V_L^{T_i}, Y^\theta)$ ($i = 1,2$) are irreducible θ-twisted V_L-modules. Note that $V_L^{T_i}$ has a θ-twisted $M(1)$-module structure. Let t_i be a basis of T_i for $i = 1,2$. Then we have a canonical θ-twisted $M(1)$-module isomorphism

$$\phi_i : M(1)(\theta) \to V_L^{T_i} : u \mapsto u \otimes t_i. \tag{2.8}$$

The action of the automorphism θ on $M(1)(\theta)$ is defined by

$$\theta(X_1(-n_1)\cdots X_m(-n_m)1) = (-1)^m X_1(-n_1)\cdots X_m(-n_m)1,$$

for $X_i \in \mathfrak{h}, n_i \in 1/2+\mathbb{N}$. Set $M(1)(\theta)^\pm$ the ±1-eigenspaces of $M(1)(\theta)$ for θ and $V_L^{T_i,\pm}$ the ±1-eigenspaces of $V_L^{T_i}$ for $\theta\otimes 1$. Then $M(1)(\theta)^\pm$ and $V_L^{T_i,\pm}$ ($i = 1,2$) become irreducible $M(1)^+$-modules and irreducible V_L^+-modules respectively (see [DN1] and [DN2] resp.). All irreducible $M(1)^+$-modules and all irreducible V_L^+-modules are classified in [DN1] and [DN2] respectively.

Theorem 2.6. (1) ([DN1]) *The set*

$$\{M(1)^{\pm}, M(1)(\theta)^{\pm}, M(1,\lambda)(\cong M(1,-\lambda))\,|\,\lambda\in\mathfrak{h}-\{0\}\,\} \tag{2.9}$$

gives all inequivalent irreducible $M(1)^+$*-modules.*
(2) ([DN2]) *The set*

$$\{V_L^{\pm}, V_{\alpha/2+L}^{\pm}, V_L^{T_i,\pm}, V_{r\alpha/2k+L}\,|\,i=1,2, 1\le r\le k-1\,\} \tag{2.10}$$

gives all inequivalent irreducible V_L^+*-modules.*

We call irreducible modules $V_L^{\pm}$, $V_{\alpha/2+L}^{\pm}$ and $V_{r\alpha/2k+L}$ *untwisted type* modules, and call $V_L^{T_i,\pm}$ $(i=1,2)$ *twisted type* modules. Here and further we write $\lambda_r = r\alpha/2k$ for $r\in\mathbb{Z}$.

2.3 Fusion rules for $M(1)^+$ and contragredient modules for V_L^+

First we list up the fusion rules for $M(1)^+$ determined in [A]. The fusion rules play central roles in determining fusion rules for V_L^+.

Theorem 2.7. ([A]) *Let* M*,* N *and* L *be irreducible* $M(1)^+$*-modules.*
(i) *If* $M=M(1)^+$*, then* $N^L_{M(1)^+N}=\delta_{N,L}$.
(ii) *If* $M=M(1)^-$*, then* $N^L_{M(1)^-N}$ *is* 0 *or* 1*, and* $N^L_{M(1)^-N}=1$ *if and only if the pair* (N,L) *is one of the following pairs:*

$$(M(1)^{\pm}, M(1)^{\mp}),\ (M(1)(\theta)^{\pm}, M(1)(\theta)^{\mp}),$$
$$(M(1,\lambda), M(1,\mu)) \text{ for } \lambda,\mu\in\mathfrak{h}-\{0\} \text{ such that } \langle\lambda,\lambda\rangle=\langle\mu,\mu\rangle.$$

(iii) *If* $M=M(1,\lambda)$ *for* $\lambda\in\mathfrak{h}-\{0\}$*, then* $N^L_{M(1,\lambda)N}$ *is* 0 *or* 1*, an* $N^L_{M(1,\lambda)N}=1$ *if and only if the pair* (N,L) *is one of the following pairs:*

$$(M(1)^{\pm}, M(1,\mu))\ (M(1,\mu), M(1)^{\pm}) \text{ for } \mu\in\mathfrak{h}-\{0\} \text{ such that } \langle\lambda,\lambda\rangle=\langle\mu,\mu\rangle,$$
$$(M(1,\mu), M(1,\nu)) \text{ for } \mu,\nu\in\mathfrak{h}-\{0\} \text{ such that } \langle\nu,\nu\rangle=\langle\lambda\pm\mu,\lambda\pm\mu\rangle,$$
$$(M(1)(\theta)^{\pm}, M(1)(\theta)^{\pm}),\ (M(1)(\theta)^{\pm}, M(1)(\theta)^{\mp}).$$

(iv) *If* $M=M(1)(\theta)^+$*, then* $N^L_{M(1)(\theta)^+N}$ *is* 0 *or* 1*, and* $N^L_{M(1)(\theta)^+N}=1$ *if and only if the pair* (N,L) *is one of the following pairs:*

$$(M(1)^{\pm}, M(1)(\theta)^{\pm}),\ (M(1)(\theta)^{\pm}, M(1)^{\pm}),$$
$$(M(1,\lambda), M(1)(\theta)^{\pm}),\ (M(1)(\theta)^{\pm}, M(1,\lambda)) \text{ for } \lambda\in\mathfrak{h}-\{0\}.$$

(v) *If $M = M(1)(\theta)^-$, then $N^L_{M(1)(\theta)^- N}$ is 0 or 1, and $N^L_{M(1)(\theta)^- N} = 1$ if and only if the pair (N, L) is one of the following pairs:*

$$(M(1)^{\pm}, M(1)(\theta)^{\mp}),\ (M(1)(\theta)^{\pm}, M(1)^{\mp}),$$
$$(M(1,\lambda), M(1)(\theta)^{\pm}),\ (M(1)(\theta)^{\pm}, M(1,\lambda)) \text{ for } \lambda \in \mathfrak{h} - \{0\}.$$

Next we discuss the contragredient modules of irreducible V_L^+-modules. We shall prove the following proposition.

Proposition 2.8. (i) *If k is even, then all irreducible V_L^+-modules are self-dual, that is , $W \cong W'$ as V_L^+-modules.*
(ii) *If k is odd, then*

$$(V^{\pm}_{\alpha/2+L})' \cong V^{\mp}_{\alpha/2+L}, (V_L^{T_1,\pm})' \cong V_L^{T_2,\pm}, (V_L^{T_2,\pm})' \cong V_L^{T_1,\pm}$$

and others are self-dual.

To prove the proposition, we use Zhu's theory (see [Z]). Let V be a vertex operator algebra. The Zhu's algebra $A(V)$ associated with V is a quotient space of V by the subspace $O(V)$ which is spanned by vectors of the form

$$a \circ b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}(a)}}{z^2} Y(a,z)b$$

for homogeneous element $a \in V$ and $b \in V$. The product of $A(V)$ is induced from the bilinear map $* : V \times V \to V$ which is defined by

$$a * b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}(a)}}{z} Y(a,z)b$$

for homogeneous element $a \in V$ and $b \in V$. Let M be an irreducible V-module. Then there is a constant $h \in \mathbb{C}$ such that M has a direct sum decomposition $M = \oplus_{n\in\mathbb{N}} M_n$, $M_n = \{\, v \in M \mid L(0)v = (h+n)v \,\}$ for $n \in \mathbb{N}$. Then the action $o(a)u = a_{\mathrm{wt}(a)-1}u$ for $a \in V$ and $u \in M$ induces an $A(V)$-module structure on M_0 which is called the top level of M, and M_0 is irreducible as $A(V)$-module. Furthermore if two irreducible V-modules M and N have top levels M_0 and N_0 which are isomorphic to each other as $A(V)$-modules, then M and N are isomorphic as V-module.

Suppose that $k \neq 1$. Then in [DN2], it is proved that the Zhu's algebra $A(V_L^+)$ is generated by three elements $[\omega]$, $[J]$ and $[E]$, where $[a]$ means the image $a + O(V_L^+)$ in $A(V_L^+)$ of $a \in V_L^+$, and $J = h(-1)^4\mathbf{1} - 2h(-3)h(-1)\mathbf{1} + (3/2)h(-2)^2\mathbf{1}$ and $E = e_\alpha + e_{-\alpha}$. Hence for an irreducible V_L^+-module M, to find the irreducible module which is isomorphic to M', it is enough to see the actions of $[\omega]$, $[J]$ and $[E]$ on the top level of M'. Since the top level of an irreducible V_L^+-module is one dimensional, they act on the top level as scalar multiple. By the construction of irreducible V_L^+-modules, we have the following table.

	V_L^+	V_L^-	V_{λ_r+L} $(1 \leq r \leq k-1)$	$V_{\alpha/2+L}^+$	$V_{\alpha/2+L}^-$
ω	0	1	$r^2/4k$	$k/4$	$k/4$
J	0	-6	$(r^2/2k)^2 - r^2/4k$	$k^4/4 - k^2/4$	$k^4/4 - k^2/4$
E	0	0	0	1	-1

	$V_L^{T_1,+}$	$V_L^{T_1,-}$	$V_L^{T_2,+}$	$V_L^{T_2,-}$
ω	$1/16$	$9/16$	$1/16$	$9/16$
J	$3/128$	$-45/128$	$3/128$	$-45/128$
E	2^{-2k+1}	$-2^{-2k+1}(4k-1)$	-2^{-2k+1}	$2^{-2k+1}(4k-1)$

Table 1. Actions of ω, J and E on the top level

Now we prove Proposition 2.8.

Proof of Proposition 2.8. Firs we consider the case $k \neq 1$. Let W be an irreducible V_L^+-module. Set the top level $W_0 = \mathbb{C}v$, and the top level of the contragredient module $W_0' = \mathbb{C}v'$. By the definition of a contragredient module, if $a \in V_L^+$ satisfies that $L(0)a = \mathrm{wt}(a)a$ and $L(1)a = 0$, we have $\langle o(a)v', v\rangle = (-1)^{\mathrm{wt}(a)}\langle v', o(a)v\rangle$, and hence

$$\langle o(\omega)v', v\rangle = \langle v', o(\omega)v\rangle, \langle o(J)v', v\rangle = \langle v', o(J)v\rangle, \langle o(E)v', v\rangle = (-1)^k\langle v', o(E)v\rangle. \quad (2.11)$$

Therefore by Table 1, we have Proposition 2.8 for $k \neq 1$.

If $k = 1$, then the dimension of the top level $(V_L^-)_0$ is two and others are one. Hence we see that $(V_L^-)' \cong V_L^-$ because the dimension of $(V_L^-)_0'$ is two. Since for irreducible V_L^+-modules except V_L^- Table 1 is valid, we may apply same arguments of the case $k \neq 1$ to such irreducible modules. Therefore Table 1 and (2.11) shows that

$$(V_L^+)' \cong V_L^+, (V_{\alpha/2+L}^{\pm})' \cong V_{\alpha/2+L}^{\mp}, (V_L^{T_1,\pm})' \cong V_L^{T_2,\pm}, (V_L^{T_2,\pm})' \cong V_L^{T_1,\pm}.$$

This proves Proposition 2.8 for $k = 1$. $\square$

3 Fusion rules for V_L^+

In Section 3.1, we give irreducible decompositions of irreducible V_L^+-modules as $M(1)^+$-modules, and prove that every fusion rules for V_L^+ are zero or one with the help of fusion rules for $M(1)^+$. The main theorem is stated in Section 3.2. The rest of sections is devoted to the proof of the theorem and it is divided into two cases; one is the case that all modules are untwisted types (Section 3.3) and the other is the case that some irreducible module is twisted type (Section 3.4).

3.1 Irreducible decompositions of irreducible V_L^+-modules as $M(1)^+$-modules

Since $V_{\lambda+L} = \bigoplus_{m\in\mathbb{Z}} M(1, \lambda + m\alpha)$ for $\lambda \in L^\circ$ and $M(1,\mu)$ is irreducible for $M(1)^+$ if $\mu \neq 0$, V_{λ_r+L} $(1 \leq r \leq k-1)$ has an irreducible decompositions for $M(1)^+$:

$$V_{\lambda_r+L} \cong \bigoplus_{m\in\mathbb{Z}} M(1, \lambda_r + m\alpha), \tag{3.1}$$

For a nonzero $\lambda \in \mathfrak{h}$, we consider the subspace $(M(1)^+ \otimes (e_\lambda \pm e_{-\lambda})) \oplus (M(1)^- \otimes (e_\lambda \mp e_{-\lambda}))$ on $M(1,\lambda) \oplus M(1,-\lambda)$. Since the action of $M(1)^+$ of $M(1,\lambda) \oplus M(1,-\lambda)$ commutes the action of θ, the subspaces $(M(1)^+ \otimes (e_\lambda \pm e_{-\lambda})) \oplus (M(1)^- \otimes (e_\lambda \mp e_{-\lambda}))$ are $M(1)^+$-submodules. In fact we have the following proposition.

Lemma 3.1. *For a nonzero $\lambda \in \mathfrak{h}$, $M(1)^+$-submodules $(M(1)^+ \otimes (e_\lambda \pm e_{-\lambda})) \oplus (M(1)^- \otimes (e_\lambda \mp e_{-\lambda}))$ of $M(1,\lambda) \oplus M(1,-\lambda)$ are isomorphic to $M(1,\lambda)$.*

Proof. Define a linear map ϕ_λ by

$$\begin{aligned} \phi_\lambda : (M(1)^+ \otimes (e_\lambda + e_{-\lambda})) \oplus (M(1)^- \otimes (e_\lambda - e_{-\lambda})) &\rightarrow M(1,\lambda) \\ u \otimes (e_\lambda + e_{-\lambda}) + v \otimes (e_\lambda - e_{-\lambda}) &\mapsto (u+v) \otimes e_\lambda, \end{aligned} \tag{3.2}$$

for $u \in M(1)^+$ and $v \in M(1)^-$. Then the linear map ϕ_λ is an injective $M(1)^+$-module homomorphism. Since $M(1,\lambda)$ is irreducible for $M(1)^+$, the homomorphism is in fact an isomorphism. Hence $M(1)^+ \otimes (e_\lambda + e_{-\lambda}) \oplus M(1)^- \otimes (e_\lambda - e_{-\lambda})$ is isomorphic to $M(1,\lambda)$ as $M(1)^+$-module. We can also prove that $M(1)^+ \otimes (e_\lambda - e_{-\lambda}) \oplus M(1)^- \otimes (e_\lambda + e_{-\lambda})$ is isomorphic to $M(1,\lambda)$ as $M(1)^+$-module in the same way. $\square$

We give irreducible decompositions of irreducible V_L^+-modules for $M(1)^+$;

Proposition 3.2. *Each irreducible V_L^+-modules decompose into direct sums of irreducible $M(1)^+$-modules as follows*;

$$\begin{aligned}
V_L^{\pm} &\cong M(1)^{\pm} \oplus \bigoplus_{m=1}^{\infty} M(1, m\alpha), && (3.3)\\
V_{\lambda_r+L} &\cong \bigoplus_{m\in\mathbb{Z}} M(1, \lambda_r + m\alpha) \text{ for } 1 \le r \le k-1, && (3.4)\\
V_{\frac{\alpha}{2}+L}^{\pm} &\cong \bigoplus_{m=0}^{\infty} M(1, \frac{\alpha}{2} + m\alpha), && (3.5)\\
V_L^{T_i,\pm} &\cong M(1)(\theta)^{\pm} \text{ for } i = 1, 2. && (3.6)
\end{aligned}$$

Proof. Irreducible decompositions of V_{λ_r+L} $(1 \le r \le k-1)$ and $V_L^{T_i,\pm}$ $(i = 1, 2)$ have already given by (3.1) and (2.8) respectively. We see that $V_L^{\pm}$ and $V_{\alpha/2+L}^{\pm}$ have direct sum decompositions

$$\begin{aligned}
V_L^{\pm} &= \bigoplus_{m=0}^{\infty}((M(1)^+ \otimes (e_{m\alpha} \pm e_{-m\alpha})) \oplus (M(1)^- \otimes (e_{m\alpha} \mp e_{-m\alpha}))),\\
V_{\frac{\alpha}{2}+L}^{\pm} &= \bigoplus_{m=0}^{\infty}((M(1)^+ \otimes (e_{\frac{\alpha}{2}+m\alpha} \pm e_{-\frac{\alpha}{2}-m\alpha})) \oplus (M(1)^- \otimes (e_{\frac{\alpha}{2}+m\alpha} \mp e_{-\frac{\alpha}{2}-m\alpha}))).
\end{aligned}$$

Hence Lemma 3.1 shows that these direct sum decompositions give irreducible decompositions of $V_L^{\pm}$ and $V_{\alpha/2+L}^{\pm}$. □

By Proposition 3.2, one see that for any irreducible V_L^+-module W, the multiplicity of an irreducible $M(1)^+$-module in W is at most one.

Using these irreducible decompositions (3.3)-(3.6), Theorem 2.7 and Corollary 2.4, we can show that all fusion rules for V_L^+ are at most one:

Proposition 3.3. *Let W^1, W^2 and W^3 be irreducible V_L^+-module.*
(1) *The fusion rule $N_{W^1W^2}^{W^3}$ is zero or one.*
(2) *If all W^i $(i = 1, 2, 3)$ are twisted type modules, then the fusion rule $N_{W^1W^2}^{W^3}$ is zero.*
(3) *If one of W^i $(i = 1, 2, 3)$ is twisted type module and others are of untwisted types, then the fusion rule $N_{W^1W^2}^{W^3}$ is zero.*

Proof. Suppose that W^1 and W^2 have irreducible $M(1)^+$-submodules M and N respectively and that W^3 has an irreducible decomposition $W^3 = \oplus_i M^i$ as $M(1)^+$-module.

By (2.2), we have an inequality

$$\dim I_{V_L^+}\left(\begin{array}{c} W^3 \\ W^1 \quad W^2 \end{array}\right) \le \sum_i \dim I_{M(1)^+}\left(\begin{array}{c} M^i \\ M \quad N \end{array}\right). \tag{3.7}$$

If W^1, W^2 and W^3 are of twisted type or if W^1 is of twisted type and W^2 and W^3 are of untwisted type, then by Theorem 2.7 (iv), (v) and (3.3)-(3.6), we see that the fusion rule for $M(1)^+$ of type $\left(\begin{array}{c} M^i \\ M \quad N \end{array}\right)$ is zero for any i. Hence (3.7) implies that the fusion rule $N_{W^1 W^2}^{W^3}$ is zero. Since the contragredient module of an (un)twisted type module is of (un)twisted type, (2) and (3) follows from Proposition 2.2.

By (2),(3) and Proposition 2.2, to show (1), it suffices to prove that if W^1 is untwisted type module and both W^2 and W^3 are of twisted types or of untwisted types, then the fusion rule $N_{W^1 W^2}^{W^3}$ is zero or one.

If W^1 is untwisted type module and W^2 and W^3 are of twisted types, then by Theorem 2.7 (i)-(iii) and irreducible decompositions (3.3)-(3.6), we see that the fusion rule for $M(1)^+$ of type $\left(\begin{array}{c} W^3 \\ M \quad W^2 \end{array}\right)$ is zero or one for any irreducible $M(1)^+$-submodules M of W^1. Hence Corollary 2.4 shows that the fusion rule $N_{W^1 W^2}^{W^3}$ is zero or one.

Now we turn to the case all W^i $(i = 1, 2, 3)$ are of untwisted types. We consider the following three cases separately; (i) $W^1 = V_L^{\pm}$, (ii) $W^1 = V_{\alpha/2+L}^{\pm}$ and (iii) $W^1 = V_{\lambda_r+L}$ for $1 \le r \le k-1$. Let $W^3 = \oplus_i M^i$ be the irreducible decomposition of W^3 for $M(1)^+$. Then it suffices to prove that the right-hand side of (3.7) is at most one for some $M(1)^+$-submodules M of W^1 and N of W^2.

(i) $W^1 = V_L^{\pm}$ cases: Take $M = M(1)^{\pm}$. By (3.3)-(3.5), we can take N to be isomorphic to $M(1,\lambda)$ for some $\lambda \in L^{\circ}$. Then by Theorem 2.7 (i) and (ii), the fusion rule for $M(1)^+$ of type $\left(\begin{array}{c} M^i \\ M \quad N \end{array}\right)$ is one if and only if M^i is isomorphic to $M(1,\lambda)$. Since the multiplicity of $M(1,\lambda)$ in the irreducible decomposition of W^3 is at most one, we see that the right-hand side of (3.7) is zero or one.

(ii) $W^1 = V_{\alpha/2+L}^{\pm}$ case: Take $M \cong M(1,\alpha/2)$. If N is isomorphic to $M(1,\lambda)$ for some $\lambda \in L^{\circ}$, then by Theorem 2.7 (iii), we see that the fusion rule for $M(1)^+$ of type $\left(\begin{array}{c} M^i \\ M \quad N \end{array}\right)$ is one if and only if M^i is isomorphic to $M(1,\lambda+\alpha/2)$ or $M(1,\lambda-\alpha/2)$. If $W^2 = V_{\alpha/2+L}^{\pm}$, then by taking $\lambda = \alpha/2$, we see that the right-hand side of (3.7) is zero unless W^3 is V_L^+ or V_L^-. So these cases and the cases $W^2 = V_L^{\pm}$ reduce to the case (i) by means of Proposition 2.2. Therefore to prove (1) in the case $W^1 = V_{\alpha/2+L}^{\pm}$, it is enough to consider

the case $W^2 = V_{\lambda_r+L}$ for some $1 \le r \le k-1$. Then by taking $\lambda = \lambda_r$, we see that the right-hand side of (3.7) is zero unless W^3 is $V_{\lambda_{k-r}+L}$. By Corollary 2.4 and Proposition 2.8, the fusion rules of types $\left(\begin{smallmatrix} & V_{\lambda_{k-r}+L} & \\ V^{\pm}_{\alpha/2+L} & & V_{\lambda_r+L} \end{smallmatrix} \right)$ is equal those of types $\left(\begin{smallmatrix} & (V^{\pm}_{\alpha/2+L})' & \\ V_{\lambda_r+L} & & V_{\lambda_{k-r}+L} \end{smallmatrix} \right)$ respectively. Hence we have to show that the right-hand side of (3.7) is at most one when $W^1 = V_{\lambda_r+L}, W^2 = V_{\lambda_{k-r}+L}$ and $W^3 = (V^{\pm}_{\alpha/2+L})'$. We take $M = M(1,\lambda_r)$ and $N = M(1,\lambda_{k-r})$. Since by Theorem 2.7 the fusion rule for $M(1)^+$ of type $\left(\begin{smallmatrix} M(1,\alpha/2+m\alpha) \\ M \quad N \end{smallmatrix} \right)$ is $\delta_{m,0}$ for $m \in \mathbb{N}$, (3.5) shows that the the right-hand side of (3.7) is at most one.

(iii) $W^1 = V_{\lambda_r+L}$ case for $1 \le r \le k-1$: By Proposition 2.2 and results of (i) and (ii), to prove (1) in the case, it is sufficient to consider the case $W^2 = V_{\lambda_s+L}$ for $1 \le s \le k-1$. Then we can take $M = M(1,\lambda_r)$ and $N = M(1,\lambda_s)$. Hence by Theorem 2.7 (iii), we see that the fusion rule for $M(1)^+$ of type $\left(\begin{smallmatrix} M^i \\ M \quad N \end{smallmatrix} \right)$ is one if and only if M^i is isomorphic to $M(1,\lambda_r+\lambda_s)$ or $M(1,\lambda_r-\lambda_s)$. By (3.3)-(3.5), one see that for $\mu,\nu \in L^{\circ}$ $M(1,\mu)$ and $M(1,\nu)$ have multiplicity one in W^3, then $\mu+\nu \in L$ or $\mu-\nu \in L$. But $(\lambda_r+\lambda_s)+(\lambda_r-\lambda_s)$ and $(\lambda_r+\lambda_s)-(\lambda_r-\lambda_s)$ are not in L. Hence by (3.3)-(3.5), we see that the right-hand side of (3.7) is zero or one. $\square$

3.2 Main Theorem

Here we give the main theorem. The proof is given in Section 3.3 and 3.4:

Theorem 3.4. *Let W^1, W^2 and W^3 be irreducible V_L^+-modules. Then* (1) *the fusion rule $N_{W^1W^2}^{W^3}$ is zero or one and* (2) *the fusion rule $N_{W^1W^2}^{W^3}$ is one if and only if W^i $(i=1,2,3)$ satisfy following cases*:
(i) *$W^1 = V_L^+$ and $W^2 \cong W^3$.*
(ii) *$W^1 = V_L^-$ and the pair (W^2, W^3) is one of the pairs*

$$(V_L^{\pm}, V_L^{\mp}),\ (V^{\pm}_{\alpha/2+L}, V^{\mp}_{\alpha/2+L}),\ (V_L^{T_1,\pm}, V_L^{T_1,\mp}),\ (V_L^{T_2,\pm}, V_L^{T_2,\mp}),$$
$$(V_{\lambda_r+L}, V_{\lambda_r+L})\ for\ 1 \le r \le k-1.$$

(iii) *$W^1 = V^+_{\alpha/2+L}$ and the pair (W^2, W^3) is one of the pairs*

$$(V_L^{\pm}, V^{\pm}_{\alpha/2+L}),\ ((V^{\pm}_{\alpha/2+L})', V_L^{\pm}),\ ((V_L^{T_1,\pm})', V_L^{T_1,\pm}),\ ((V_L^{T_2,\pm})', V_L^{T_2,\mp}),$$
$$(V_{\lambda_r+L}, V_{\alpha/2-\lambda_r+L})\ for\ 1 \le r \le k-1.$$

(iv) $W^1 = V^-_{\alpha/2+L}$ *and the pair* (W^2, W^3) *is one of the pairs*

$$(V_L^{\pm}, V^{\mp}_{\alpha/2+L}),\ ((V^{\pm}_{\alpha/2+L})', V_L^{\mp}),\ ((V_L^{T_1,\pm})', V_L^{T_1,\mp}),\ ((V_L^{T_2,\pm})', V_L^{T_2,\pm}),$$
$$(V_{\lambda_r+L}, V_{\alpha/2-\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1.$$

(v) $W^1 = V_{\lambda_r+L}$ *for* $1 \le r \le k-1$ *and the pair* (W^2, W^3) *is one of the pairs*

$$(V_L^{\pm}, V_{\lambda_r+L}),\ (V_L^{\pm}, V_{\lambda_r+L}),\ (V^{\pm}_{\alpha/2+L}, V_{\alpha/2-\lambda_r+L}),\ (V_{\alpha/2-\lambda_r+L}, V^{\pm}_{\alpha/2+L}),$$
$$(V_{\lambda_s+L}, V_{\lambda_r\pm\lambda_s+L})\ \textit{for}\ 1 \le s \le k-1,$$
$$(V_L^{T_1,\pm}, V_L^{T_1,\pm}),\ (V_L^{T_1,\pm}, V_L^{T_1,\mp}),\ (V_L^{T_2,\pm}, V_L^{T_2,\pm}),\ (V_L^{T_2,\pm}, V_L^{T_2,\mp})\ \textit{if}\ r\ \textit{is even},$$
$$(V_L^{T_1,\pm}, V_L^{T_2,\pm}),\ (V_L^{T_1,\pm}, V_L^{T_2,\mp}),\ (V_L^{T_2,\pm}, V_L^{T_1,\pm}),\ (V_L^{T_2,\pm}, V_L^{T_1,\mp})\ \textit{if}\ r\ \textit{is odd}.$$

(vi) $W^1 = (V_L^{T_1,+})'$ *and the pair* (W^2, W^3) *is one of the pairs*

$$(V_L^{\pm}, (V_L^{T_1,\pm})'),\ (V_L^{T_1,\pm}, V_L^{\pm}),\ (V^{\pm}_{\alpha/2+L}, V_L^{T_1,\pm}),\ ((V_L^{T_1,\pm})', (V^{\pm}_{\alpha/2+L})'),$$
$$(V_{\lambda_r+L}, (V_L^{T_1,\pm})')\ \textit{and}\ (V_L^{T_1,\pm}, V_{\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1\ \textit{and}\ r\ \textit{is even},$$
$$(V_{\lambda_r+L}, (V_L^{T_2,\pm})')\ \textit{and}\ (V_L^{T_2,\pm}, V_{\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1\ \textit{and}\ r\ \textit{is odd}.$$

(vii) $W^1 = (V_L^{T_1,-})'$ *and the pair* (W^2, W^3) *is one of the pairs*

$$(V_L^{\pm}, (V_L^{T_1,\mp})'),\ (V_L^{T_1,\pm}, V_L^{\mp}),\ (V^{\pm}_{\alpha/2+L}, V_L^{T_1,\mp}),\ ((V_L^{T_1,\pm})', (V^{\mp}_{\alpha/2+L})'),$$
$$(V_{\lambda_r+L}, (V_L^{T_1,\pm})')\ \textit{and}\ (V_L^{T_1,\pm}, V_{\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1\ \textit{and}\ r\ \textit{is even},$$
$$(V_{\lambda_r+L}, (V_L^{T_2,\pm})')\ \textit{and}\ (V_L^{T_2,\pm}, V_{\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1\ \textit{and}\ r\ \textit{is odd}.$$

(viii) $W^1 = (V_L^{T_2,+})'$ *and the pair* (W^2, W^3) *is one of the pairs*

$$(V_L^{\pm}, (V_L^{T_2,\pm})'),\ (V_L^{T_2,\pm}, V_L^{\pm}),\ (V^{\pm}_{\alpha/2+L}, V_L^{T_2,\mp}),\ ((V_L^{T_2,\pm})', (V^{\mp}_{\alpha/2+L})'),$$
$$(V_{\lambda_r+L}, (V_L^{T_2,\pm})')\ \textit{and}\ (V_L^{T_2,\pm}, V_{\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1\ \textit{and}\ r\ \textit{is even},$$
$$(V_{\lambda_r+L}, (V_L^{T_1,\pm})')\ \textit{and}\ (V_L^{T_1,\pm}, V_{\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1\ \textit{and}\ r\ \textit{is odd}.$$

(ix) $W^1 = (V_L^{T_2,-})'$ *and the pair* (W^2, W^3) *is one of the pairs*

$$(V_L^{\pm}, (V_L^{T_2,\mp})'),\ (V_L^{T_2,\pm}, V_L^{\mp}),\ (V^{\pm}_{\alpha/2+L}, V_L^{T_2,\pm}),\ ((V_L^{T_2,\pm})', (V^{\pm}_{\alpha/2+L})'),$$
$$(V_{\lambda_r+L}, (V_L^{T_2,\pm})')\ \textit{and}\ (V_L^{T_2,\pm}, V_{\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1\ \textit{and}\ r\ \textit{is even},$$
$$(V_{\lambda_r+L}, (V_L^{T_1,\pm})')\ \textit{and}\ (V_L^{T_1,\pm}, V_{\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1\ \textit{and}\ r\ \textit{is odd}.$$

Since (1) of the main theorem has already proved in Proposition 3.3, to prove the theorem, it is enough to show that for irreducible V_L^+-modules W^1, W^2 and W^3, the fusion rule $N_{W^1 W^2}^{W^3}$ is nonzero if and only if the triple (W^1, W^2, W^3) satisfy indicated cases in the theorem. In Section 3.3, we prove this in the case all W^i $(i = 1, 2, 3)$ are untwisted type modules, and in Section 3.4 we do in the case one of W^i $(i = 1, 2, 3)$ is twisted type modules. To show "if" part, we shall construct nonzero intertwining operators explicitly.

3.3 Fusion rules for untwisted type modules

We construct nonzero intertwining operators for untwisted type modules. For this purpose, we review intertwining operators for V_L following [DL].

Let $\lambda, \mu \in L^\circ$. An intertwining operator of type $\binom{V_{\lambda+\mu+L}}{V_{\lambda+L}\ \ V_{\mu+L}}$ is constructed as follows: As shown in Chapter 8 of [FLM], the operator $\mathcal{Y}^\circ$ satisfies Jacobi identity and $L(-1)$-derivative property on V_{L° for $\beta \in L$, $\lambda \in L^\circ, a \in M(1,\beta)$ and $u \in M(1,\lambda)$:

$$
\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1)\mathcal{Y}^\circ(u, z_2) - (-1)^{\langle\beta,\lambda\rangle} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{Y}^\circ(u, z_2)Y(a, z_1)\\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{Y}^\circ(Y(a, z_0)u, z_2),
\end{aligned}
$$

$$
\frac{d}{dz}\mathcal{Y}^\circ(u, z) = \mathcal{Y}^\circ(L(-1)u, z).
$$

Let π_λ $(\lambda \in L^\circ)$ be the linear endomorphism of V_{L° defined by $\pi_\lambda(v) = e^{\langle\lambda,\mu\rangle\pi i}\, v$ for $\mu \in L^\circ$ and $v \in M(1,\mu)$. Set $\mathcal{Y}_{r,s}(u, z) = \mathcal{Y}^\circ(u, z)\pi_{\lambda_r}|_{V_{\lambda_s+L}}$ for $r, s \in \mathbb{Z}$ and $u \in V_{\lambda_r+L}$. Then the operator $\mathcal{Y}_{r,s}$ gives a nonzero intertwining operator for V_L of type $\binom{V_{\lambda_r+\lambda_s+L}}{V_{\lambda_r+L}\ \ V_{\lambda_s+L}}$ (see [DL]).

Proposition 3.5. *Fusion rules of the following types are nonzero;*

(i) $\binom{V_{(\lambda_r\pm\lambda_s)+L}}{V_{\lambda_r+L}\ \ V_{\lambda_s+L}}$ *for* $1 \le r, s \le k-1$,

(ii) $\binom{V_L^\pm}{V_L^+\ \ V_L^\pm}, \binom{V_L^\mp}{V_L^-\ \ V_L^\pm}$ *and* $\binom{V_{\lambda_r+L}}{V_L^\pm\ \ V_{\lambda_r+L}}$ *for* $0 \le r \le k-1$,

(iii) $\binom{V_{\alpha/2+L}^\pm}{V_L^+\ \ V_{\alpha/2+L}^\pm}, \binom{V_{\alpha/2+L}^\mp}{V_L^-\ \ V_{\alpha/2+L}^\pm}$ *and* $\binom{V_{(\alpha/2-\lambda_r)+L}}{V_{\alpha/2+L}^\pm\ \ V_{\lambda_r+L}}$ *for* $0 \le r \le k-1$.

Proof. Since $(V_L, Y), (V_{\alpha/2+L}, Y)$ and (V_{λ_r+L}, Y) $(1 \le r \le k-1)$ are irreducible V_L-modules, the vertex operator Y gives nonzero intertwining operators for V_L of types

$$\left({V_L \atop V_L \;\; V_L} \right), \left({V_{\alpha/2+L} \atop V_L \;\; V_{\alpha/2+L}} \right) \text{ and } \left({V_{\lambda_r+L} \atop V_L \;\; V_{\lambda_r+L}} \right)$$

for $1 \le r \le k-1$. Hence $Y(a,z)u$ is nonzero for any nonzero $a \in V_L$ and nonzero $u \in V_{\lambda_s+L}$ $(s \in \mathbb{Z})$ by Corollary 2.4. Therefore since $\theta Y(a,z)\theta = Y(\theta(a), z)$ for $a \in V_L$, Y gives nonzero intertwining operators for V_L^+ of types

$$\left({V_L^{\pm} \atop V_L^+ \;\; V_L^{\pm}} \right), \left({V_L^{\mp} \atop V_L^- \;\; V_L^{\pm}} \right), \left({V_{\alpha/2+L}^{\pm} \atop V_L^+ \;\; V_{\alpha/2+L}^{\pm}} \right), \left({V_{\alpha/2+L}^{\mp} \atop V_L^- \;\; V_{\alpha/2+L}^{\pm}} \right) \text{ and } \left({V_{\lambda_r+L} \atop V_L^{\pm} \;\; V_{\lambda_r+L}} \right)$$

for $1 \le r \le k-1$.

Next we show that fusion rules of types $\left({V_{(\lambda_r \pm \lambda_s)+L} \atop V_{\lambda_r+L} \;\; V_{\lambda_s+L}} \right)$ for $r, s \in \mathbb{Z}$ are nonzero. Define $\mathcal{Y}_{r,-s} \circ \theta$ by $(\mathcal{Y}_{r,-s} \circ \theta)(u,z)v = \mathcal{Y}_{r,-s}(u,z)\theta(v)$ for $u \in V_{\lambda_r+L}$ and $v \in V_{\lambda_s+L}$. Then $\mathcal{Y}_{r,s}$ is a nonzero intertwining operator for V_L^+ of type $\left({V_{(\lambda_r - \lambda_s)+L} \atop V_{\lambda_r+L} \;\; V_{\lambda_s+L}} \right)$ since θ commutes the action of V_L^+. This proves that fusion rules of types $\left({V_{(\lambda_r \pm \lambda_s)+L} \atop V_{\lambda_r+L} \;\; V_{\lambda_s+L}} \right)$ are nonzero for any $r, s \in \mathbb{Z}$.

Finally we show that fusion rule of type $\left({V_{(\alpha/2-\lambda_r)+L} \atop V_{\alpha/2+L}^{\pm} \;\; V_{\lambda_r+L}} \right)$ for $r \in \mathbb{Z}$ are nonzero. Since $V_{\alpha/2+L}$ is an irreducible V_L-module for any $r \in \mathbb{Z}$, Corollary 2.4 shows that $\mathcal{Y}_{k,r}(u,z)v$ is nonzero for any nonzero $u \in V_{\alpha/2+L}$ and nonzero $v \in V_{\lambda_r+L}$. Hence $(\mathcal{Y}_{k,-r} \circ \theta)(u,z)v$ is also nonzero for any nonzero $u \in V_{\alpha/2+L}$ and nonzero $v \in V_{\lambda_r+L}$. Therefore $\mathcal{Y}_{k,-r} \circ \theta$ gives nonzero intertwining operators of types $\left({V_{(\alpha/2-\lambda_r)+L} \atop V_{\alpha/2+L}^{\pm} \;\; V_{\lambda_r+L}} \right)$. $\square$

Next we show that for untwisted type modules W^i $(i = 1, 2, 3)$, if the fusion rule $N_{W^1 W^2}^{W^3}$ is nonzero, then the type $\left({W^3 \atop W^1 \;\; W^2} \right)$ is given from types in Proposition 3.5 by using Proposition 2.3. For this it suffices to prove the following proposition.

Proposition 3.6. *Let W^1, W^2 and W^3 be untwisted type modules. Then the fusion rule $N_{W^1 W^2}^{W^3}$ is zero if W^i $(i = 1, 2, 3)$ satisfy following cases:*

(i) $W^1 = V_L^+$, *and W^2, W^3 are inequivalent.*

(ii) $W^1 = V_L^-$ *and the pair (W^1, W^2) is one of following pairs*

$$\begin{aligned} (W^2, W^3) \;=\; & (V_L^-, V_L^-),\ (V_{\alpha/2+L}^{\pm}, V_{\alpha/2+L}^{\pm}), \\ & (V_{\lambda_r+L}, V_{\lambda_s+L}) \text{ for } 1 \le r, s \le k-1 \text{ and } \lambda_r \ne \lambda_s, \\ & (V_L^-, V_{\lambda_r+L}),\ (V_{\alpha/2+L}^{\pm}, V_{\lambda_r+L}) \text{ for } 1 \le r \le k-1. \end{aligned}$$

(iii) $W^3 = V^{\pm}_{\alpha/2+L}$ *and the pair* (W^1, W^2) *is one of following pairs*

$$\begin{aligned}(W^1, W^2) &= (V_{\lambda_r+L}, V_{\lambda_s+L}) \text{ for } 1 \le r, s \le k-1 \text{ and } \lambda_r + \lambda_s \ne \alpha/2,\\ &\quad (V^{\pm}_{\alpha/2+L}, V_{\lambda_r+L}) \text{ and } 1 \le r \le k-1.\end{aligned}$$

(iv) $W^1 = V_{\lambda_r+L}$ *for* $1 \le r \le k-1$ *and the pair* (W^1, W^2) *is one of following pairs*

$$(W^1, W^2) = (V_{\lambda_s+L}, V_{\lambda_t+L}) \text{ for } 1 \le s, t \le k-1 \text{ and } \lambda_t \ne \lambda_r \pm \lambda_s \text{ and } \lambda_s - \lambda_r.$$

Proof. Lemma 2.5 proves the proposition in the case (i).

Next we consider the cases (ii) except the pair $(V^{\pm}_{\alpha/2+L}, V^{\pm}_{\alpha/2+L})$, (iii) and (iv). Let $W^3 = \oplus_i M^i$ be the irreducible decomposition of W^3 for $M(1)^+$. Then we can find irreducible $M(1)^+$-submodules M of W^1 and N of W^3 such that the fusion rule for $M(1)^+$ of type $\binom{M^i}{M \ \ N}$ is zero. Hence (3.7) implies that the fusion rule $N^{W^3}_{W^1W^2}$ is zero; for example, in the case $W^1 = W^2 = W^3 = V_L^-$, we take $M = N = M(1)^-$, etc..

It remains to prove the proposition in the cases $(W^1, W^2) = (V^{\pm}_{\alpha/2+L}, V^{\pm}_{\alpha/2+L})$ of (ii). We prove that the fusion rule of type $\binom{V^+_{\alpha/2+L}}{V_L^- \ \ V^+_{\alpha/2+L}}$ is zero. The case of type $\binom{V^-_{\alpha/2+L}}{V_L^- \ \ V^-_{\alpha/2+L}}$ can be also proved in the similar way.

Set

$$V^+_{\alpha/2+L}[m] = M(1)^+ \otimes (e_{\frac{\alpha}{2}+m\alpha} + e_{-(\frac{\alpha}{2}+m\alpha)}) \oplus M(1)^- \otimes (e_{\frac{\alpha}{2}+m\alpha} - e_{-(\frac{\alpha}{2}+m\alpha)}), \tag{3.8}$$

for $m \in \mathbb{N}$. Note that $V^+_{\alpha/2+L}[m]$ is isomorphic to $M(1, \alpha/2 + m\alpha)$ as $M(1)^+$-module by Proposition 3.1. Let $\mathcal{Y}$ be an intertwining operator of type $\binom{V^+_{\alpha/2+L}}{V_L^- \ \ V^+_{\alpha/2+L}}$. By Theorem 2.7 (ii), we have $\mathcal{Y}(u, z)v \in V^+_{\alpha/2+L}[0]((z))$ for $u \in M(1)^-$ and $v \in V^+_{\alpha/2+L}[0]$. Let $\phi_{\alpha/2} : V^+_{\alpha/2+L}[0] \to M(1, \alpha/2)$ be the $M(1)^+$-module isomorphism defined in (3.2). For simplicity, we denote $\phi = \phi_{\alpha/2}$. Then the operator $\phi \circ \mathcal{Y} \circ \phi^{-1}$ defined by $(\phi \circ \mathcal{Y} \circ \phi^{-1})(u, z)v = \phi\mathcal{Y}(u, z)\phi^{-1}(v)$ for $u \in M(1)^-$ and $v \in M(1, \alpha/2)$ gives an intertwining operator of type $\binom{M(1,\alpha/2)}{M(1)^- \ \ M(1,\alpha/2)}$. Since the dimension of $I_{M(1)^+}\binom{M(1,\alpha/2)}{M(1)^- \ \ M(1,\alpha/2)}$ is one and the corresponding intertwining operator is given by a scalar multiple of the vertex operator Y of the $M(1)$-module $(M(1, \alpha/2), Y)$, there exists a constant $d \in \mathbb{C}$ such that

$$\mathcal{Y}(u, z)v = d\,\phi^{-1}Y(u, z)\phi(v)$$

for every $u \in M(1)^-$ and $v \in V^+_{\alpha/2+L}[0]$. We write $\mathcal{Y}(u,z) = \sum_{n\in\mathbb{Z}} \tilde{u}(n)z^{-n-1}$ $\tilde{u} \in \operatorname{End} V^+_{\alpha/2+L}$ for $u \in V_L^-$. Take $u = h(-1)\mathbf{1}$ and $v = e_{\alpha/2} + e_{-\alpha/2}$, then we have

$$\begin{aligned} \tilde{h}(0)(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}) &= d\langle h, \frac{\alpha}{2}\rangle(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), && (3.9)\\ \tilde{h}(-1)(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}) &= d(h(-1)e_{\frac{\alpha}{2}} - h(-1)e_{-\frac{\alpha}{2}}), && (3.10)\end{aligned}$$

where we denote $(\widetilde{h(-1)\mathbf{1}})(n)$ by $\tilde{h}(n)$ for $n \in \mathbb{Z}$. By direct culculations, we see that

$$E_{k-1}(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}) = (e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), \qquad (3.11)$$

$$E_k(h(-1)e_{\frac{\alpha}{2}} - h(-1)e_{-\frac{\alpha}{2}}) = \langle h, \alpha\rangle(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), \qquad (3.12)$$

where $E = e_\alpha + e_{-\alpha} \in V_L^+$. Let $F = e_\alpha - e_{-\alpha} \in V_L^-$. Then by Jacobi identity, we have a commutation relation

$$[E_m, \tilde{h}(n)] = -\langle h, \alpha\rangle \tilde{F}(m+n) \qquad (3.13)$$

for $m, n \in \mathbb{Z}$. Hence (3.9) and (3.11) imply that $\tilde{F}(k-1)(e_{\alpha/2} + e_{-\alpha/2}) = 0$ (take $m = k-1, n = 0$ in (3.13)). On the other hand, by (3.10) and (3.12) we have

$$\begin{aligned} -\langle h, \alpha\rangle \tilde{F}(k-1)(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}) &= [E_k, \tilde{h}(-1)](e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}})\\ &= d\langle h, \alpha\rangle(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}),\end{aligned}$$

(take $m = k, n = -1$ in (3.13)). Therefore $d = 0$. This implies that $\mathcal{Y}(h(-1)\mathbf{1}, z)(e_{\alpha/2} + e_{-\alpha/2}) = 0$, and then Lemma 2.3 shows $\mathcal{Y} = 0$. Thus the fusion rule of type $\binom{V^+_{\alpha/2+L}}{V_L^- \quad V^+_{\alpha/2+L}}$ is zero. $\square$

Consequently, by Proposition 2.2, Proposition 3.6, Proposition 2.8, Proposition 3.5 and Proposition 3.3, we can determine fusion rules for untwisted type modules.

Proposition 3.7. *Let W^1, W^2 and W^3 be untwisted type V_L^+-modules. Then the fusion rule $N^{W^3}_{W^1W^2}$ is zero or one. The fusion rule $N^{W^3}_{W^1W^2}$ is one if and only if W^i $(i = 1,2,3)$ satisfy the following cases*:

(i) $W^1 = V_L^+$ *and* $W^2 \cong W^3$.

(ii) $W^1 = V_L^-$ *and the pair* (W^2, W^3) *is one of pairs*

$$(V_L^\pm, V_L^\mp),\ (V^\pm_{\alpha/2+L}, V^\mp_{\alpha/2+L}),\ (V_{\lambda_r+L}, V_{\lambda_r+L})\ \mathit{for}\ 1 \le r \le k-1.$$

(iii) $W^1 = V_{\alpha/2+L}^{+}$ *and the pair* (W^2, W^3) *is one of pairs*

$$(V_L^{\pm}, V_{\alpha/2+L}^{\pm}),\ ((V_{\alpha/2+L}^{\pm})', V_L^{\pm}),\ (V_{\lambda_r+L}, V_{\alpha/2-\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1.$$

(iv) $W^1 = V_{\alpha/2+L}^{-}$ *and the pair* (W^2, W^3) *is one of pairs*

$$(V_L^{\pm}, V_{\alpha/2+L}^{\mp}),\ ((V_{\alpha/2+L}^{\pm})', V_L^{\mp}),\ (V_{\lambda_r+L}, V_{\alpha/2-\lambda_r+L})\ \textit{for}\ 1 \le r \le k-1.$$

(v) $W^1 = V_{\lambda_r+L}$ *for* $1 \le r \le k-1$ *and the pair* (W^2, W^3) *is one of pairs*

$$\begin{aligned}&(V_L^{\pm}, V_L^{\pm}),\ (V_{\alpha/2+L}^{\pm}, V_{\alpha/2-\lambda_r+L}),\ (V_{\alpha/2-\lambda_r+L}, V_{\alpha/2+L}^{\pm}),\\ &(V_{\lambda_s+L}, V_{\lambda_r\pm\lambda_s+L})\ \textit{for}\ 1 \le s \le k-1.\end{aligned}$$

3.4 Fusion rules involving twisted type modules

Set $\mathcal{P}_L = L^{\circ} \times \{1,2\} \times \{1,2\}$. We call $(\lambda, i, j) \in \mathcal{P}_L$ a *quasi-admissible triple* if λ, i and j satisfies

$$(-1)^{\langle\lambda,\alpha\rangle+\delta_{i,j}+1} = 1.$$

We denote the set of all quasi-admissible triples by $\mathcal{Q}_L$. For a quasi-admissible triple $(\lambda, i, j) \in \mathcal{Q}_L$, we first construct an intertwining operator for V_L^+ of type $\left({V_L^{T_j} \atop V_{\lambda+L}\ \ V_L^{T_i}} \right)$.

As shown in Chapter 9 of [FLM] the operator $\mathcal{Y}^{\theta}$ satisfies twisted Jacobi identity and $L(-1)$-derivative property

$$\begin{aligned}&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{Y}^{\theta}(a,z_1)\mathcal{Y}^{\theta}(u,z_2) - (-1)^{\langle\beta,\lambda\rangle} z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\mathcal{Y}^{\theta}(u,z_2)\mathcal{Y}^{\theta}(a,z_1)\\ &= \frac{1}{2}\sum_{p=0,1} z_2^{-1}\delta\left((-1)^p\frac{(z_1-z_0)^{1/2}}{z_2^{1/2}}\right)\mathcal{Y}^{\theta}(Y(\theta^p(a),z_0)u,z_2),\end{aligned} \tag{3.14}$$

and

$$\frac{d}{dz}\mathcal{Y}^{\theta}(u,z) = \mathcal{Y}^{\theta}(L(-1)u,z) \tag{3.15}$$

for $\beta \in L, \lambda \in L^{\circ}, a \in M(1,\beta)$ and $u \in M(1,\lambda)$. Then we have following lemma.

Lemma 3.8. (1) *The intertwining operator* $\mathcal{Y}^{\theta}$ *give nonzero intertwining operators of types*

$$\left({M(1)(\theta)^{\pm} \atop M(1,\lambda)\ \ M(1)(\theta)^{\pm}} \right), \left({M(1)(\theta)^{\mp} \atop M(1,\lambda)\ \ M(1)(\theta)^{\pm}} \right)\ \textit{for}\ \lambda \in L^{\circ}.$$

(2) *Define* $\mathcal{Y}^\theta \circ \theta$ *by* $(\mathcal{Y}^\theta \circ \theta)(u,z) = \mathcal{Y}^\theta(\theta(u), z)$ *for* $u \in V_{L^\circ}$*. Then* $\mathcal{Y}^\theta \circ \theta$ *gives nonzero intertwining operators for* $M(1)^+$ *of types* $\binom{M(1)(\theta)}{M(1,\lambda)\ \ M(1)(\theta)^\pm}$*. Moreover restrictions of* $\mathcal{Y}^\theta$ *and* $\mathcal{Y}^\theta \circ \theta$ *to* $M(1,\lambda) \otimes M(1)(\theta)^\pm$ *form a basis of the vector space* $I\binom{M(1)(\theta)}{M(1,\lambda)\ \ M(1)(\theta)^\pm}$ *respectively.*

Proof. (1) is proved in [A, Proposition 4.4]. Clearly $\mathcal{Y}^\theta \circ \theta$ gives nonzero intertwining operators of types $\binom{M(1)(\theta)}{M(1,\lambda)\ \ M(1)(\theta)^\pm}$. Now we show the second assertion of (2). Since $\theta \mathcal{Y}^\theta(u,z)\theta(v) = \mathcal{Y}^\theta(\theta(u), z)v$ for $u \in M(1,\lambda)$ and $v \in M(1)(\theta)^\pm$, we have

$$p_\pm((\mathcal{Y}^\theta \circ \theta)(u,z)v) = \pm p_\pm(\mathcal{Y}^\theta(u,z)\theta(v)),$$

where $p_\pm$ is the canonical projection from $M(1)(\theta)$ to $M(1)(\theta)^\pm$ respectively. Hence by Lemma 2.3 and (1), we see that $\mathcal{Y}^\theta$ and $\mathcal{Y}^\theta \circ \theta$ are linearly independent in the vector spaces $I\binom{M(1)(\theta)}{M(1,\lambda)\ \ M(1)(\theta)^\pm}$. Since the fusion rules of types $\binom{M(1)(\theta)}{M(1,\lambda)\ \ M(1)(\theta)^\pm}$ are two by Theorem 3.4, $\mathcal{Y}^\theta$ and $\mathcal{Y}^\theta \circ \theta$ in fact form a basis of $I\binom{M(1)(\theta)}{M(1,\lambda)\ \ M(1)(\theta)^\pm}$. This proves (2). $\square$

Set $T = T^1 \oplus T^2$ the direct sum of the irreducible $\mathbb{C}[L]$-modules T^1 and T^2, and define a linear endomorphism $\psi \in \operatorname{End} T$ by $\psi(t_1) = t_2, \psi(t_2) = t_1$, where t_i is a basis of T^i for $i = 1, 2$. For $\lambda \in L^\circ$, we write $\lambda = r\alpha/2k + m\alpha$ for $-k+1 \le r \le k$ and $m \in \mathbb{Z}$, and define $\psi_\lambda \in \operatorname{End} T$ by

$$\psi_\lambda = e_{m\alpha} \circ \underbrace{\psi \circ \cdots \circ \psi}_{r}.$$

Set $\tilde{\mathcal{Y}}(u,z) = \mathcal{Y}^\theta(u,z) \otimes \psi_\lambda$ for $\lambda \in L^\circ$ and $u \in M(1,\lambda)$, and extend it to V_{L° by linearity. Then we have following proposition.

Proposition 3.9. (1) *For* $\lambda \in L^\circ$*, the linear map* ψ_λ *has following properties:*

$$e_\beta \circ \psi_\lambda = (-1)^{\langle \beta, \lambda \rangle} \psi_\lambda \circ e_\beta = \psi_{\lambda+\beta} \text{ for all } \beta \in L.$$

(2) *For* $a \in V_L$ *and* $u \in V_{\lambda+L}$*, we have*

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y^\theta(a, z_1)\tilde{\mathcal{Y}}(u, z_2) - \delta\left(\frac{z_2 - z_1}{-z_0}\right) \tilde{\mathcal{Y}}(u, z_2) Y^\theta(a, z_1) \\ &= \frac{1}{2}\sum_{p=0,1} z_2^{-1}\delta\left((-1)^p \frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}}\right) \tilde{\mathcal{Y}}(Y(\theta^p(a), z_0)u, z_2) \end{aligned}$$

and

$$\frac{d}{dz}\tilde{\mathcal{Y}}(u,z)=\tilde{\mathcal{Y}}(L(-1)u,z).$$

Proof. Since $e_\alpha\circ\psi=-\psi\circ e_\alpha$, we have $e_{m\alpha}\circ\psi^r=(-1)^{mr}\psi^r\circ e_{m\alpha}$ for $m,r\in\mathbb{Z}$. Therefore ψ_λ ($\lambda\in L^\circ$) satisfies $e_\beta\circ\psi_\lambda=(-1)^{\langle\beta,\lambda\rangle}\psi_\lambda\circ e_\beta$ and $e_\beta\circ\psi_\lambda=\psi_{\lambda+\beta}$ for $\beta\in L$. This proves (1). Then (2) follows from (3.14), (3.15) and (1). □

We note that for every quasi-admissible triple $(\lambda,i,j)\in\mathcal{Q}_L$, $\psi_\lambda(T^i)=T^j$. Thus we have

Proposition 3.10. *Let* $(\lambda,i,j)\in\mathcal{Q}_L$ *be an admissible triple. The restriction of* $\tilde{\mathcal{Y}}$ *to* $V_{\lambda+L}\otimes V_L^{T_i}$ *gives an intertwining operator for* V_L^+ *of type* $\begin{pmatrix} & V_L^{T_j} & \\ V_{\lambda+L} & & V_L^{T_i}\end{pmatrix}$.

Now we have some nonzero intertwining operators by restricting $\tilde{\mathcal{Y}}$ to irreducible V_L^+-modules.

Proposition 3.11. *Fusion rules of following types are nonzero*;

(i) $\begin{pmatrix} & V_L^{T_j,\pm} & \\ V_{\lambda_r+L} & & V_L^{T_i,\pm}\end{pmatrix}$, $\begin{pmatrix} & V_L^{T_j,\mp} & \\ V_{\lambda_r+L} & & V_L^{T_i,\pm}\end{pmatrix}$ *for* $r\in\mathbb{Z}$ *and* $(\lambda_r,i,j)\in\mathcal{Q}_L$,

(ii) $\begin{pmatrix} & V_L^{T_i,\pm} & \\ V_L^+ & & V_L^{T_i,\pm}\end{pmatrix}$, $\begin{pmatrix} & V_L^{T_i,\mp} & \\ V_L^- & & V_L^{T_i,\pm}\end{pmatrix}$ *for* $i\in\{1,2\}$,

(iii) $\begin{pmatrix} & V_L^{T_1,\pm} & \\ V_{\alpha/2+L}^+ & & (V_L^{T_1,\pm})'\end{pmatrix}$, $\begin{pmatrix} & V_L^{T_2,\mp} & \\ V_{\alpha/2+L}^+ & & (V_L^{T_2,\pm})'\end{pmatrix}$, $\begin{pmatrix} & V_L^{T_1,\mp} & \\ V_{\alpha/2+L}^- & & (V_L^{T_1,\pm})'\end{pmatrix}$, $\begin{pmatrix} & V_L^{T_2,\pm} & \\ V_{\alpha/2+L}^- & & (V_L^{T_2,\pm})'\end{pmatrix}$.

Proof. By Lemma 3.8 and Proposition 3.10, we see that $\tilde{\mathcal{Y}}$ gives nonzero intertwining operator of types $\begin{pmatrix} & V_L^{T_j,\pm} & \\ V_{\lambda_r+L} & & V_L^{T_i,\pm}\end{pmatrix}$ and $\begin{pmatrix} & V_L^{T_j,\mp} & \\ V_{\lambda_r+L} & & V_L^{T_i,\pm}\end{pmatrix}$ for $r\in\mathbb{Z}$ and $(\lambda_r,i,j)\in\mathcal{Q}_L$.

Next we shows that fusion rules of types in (ii) and (iii) are nonzero. By Lemma 3.8 and Corollary 2.4, $\mathcal{Y}^\theta(u\pm\theta(u),z)v=(\mathcal{Y}^\theta\pm\mathcal{Y}^\theta\circ\theta)(u,z)v$ are nonzero for any nonzero $u\in M(1,\lambda)$ ($\lambda\in L^\circ$) and nonzero $v\in M(1)(\theta)^\pm$. Thus by Proposition 3.10, we see that $\tilde{\mathcal{Y}}$ give nonzero intertwining operators of types

$$\begin{pmatrix} & V_L^{T_i} & \\ V_L^+ & & V_L^{T_i,\pm}\end{pmatrix},\begin{pmatrix} & V_L^{T_i} & \\ V_L^- & & V_L^{T_i,\pm}\end{pmatrix},\begin{pmatrix} & V_L^{T_i} & \\ V_{\alpha/2+L}^+ & & (V_L^{T_i,\pm})'\end{pmatrix},\begin{pmatrix} & V_L^{T_i} & \\ V_{\alpha/2+L}^- & & (V_L^{T_i,\pm})'\end{pmatrix}\text{ for }i\in\{1,2\}.$$

By the definition of ψ_λ ($\lambda\in L^\circ$), we have

$$\psi_{-m\alpha}=\psi_{m\alpha},\ \psi_{-(\alpha/2+m\alpha)}=e_{-\alpha}\psi_{\alpha/2+m\alpha}\text{ for }m\in\mathbb{Z}.\tag{3.16}$$

Since $\theta\tilde{\mathcal{Y}}(u,z)\theta = \mathcal{Y}^{\theta}(\theta(u),z)\otimes\psi_\lambda$ for $\lambda\in L^\circ$ and $u\in M(1,\lambda)$, by (3.16) we have

$$\theta\tilde{\mathcal{Y}}(u,z)\theta = \tilde{\mathcal{Y}}(\theta(u),z) \text{ for } u\in V_L,$$
$$\theta\tilde{\mathcal{Y}}(u,z)\theta = e_\alpha\tilde{\mathcal{Y}}(\theta(u),z) \text{ for } u\in V_{\alpha/2+L}.$$

This proves that $\tilde{\mathcal{Y}}$ gives nonzero intertwining operators of types indicated in (ii) and (iii) of the proposition; for instance, for $u\in V^+_{\alpha/2+L}$ and $v\in (V_L^{T_2,-})'$, we have

$$\begin{aligned}\theta\tilde{\mathcal{Y}}(u,z)v &= e_\alpha\tilde{\mathcal{Y}}(\theta(u),z)\theta(v)\\ &= \tilde{\mathcal{Y}}(u,z)v.\end{aligned}$$

Hence $\tilde{\mathcal{Y}}(u,z)v \in V_L^{T_2,+}\{z\}$. Thus $\tilde{\mathcal{Y}}$ gives a nonzero intertwining operator of type $\left(\begin{smallmatrix} & V_L^{T_2,+} & \\ V^+_{\alpha/2+L} & & (V_L^{T_2,-})'\end{smallmatrix}\right)$. $\square$

We shall show the following proposition. The proof is given after Proposition 3.13.

Proposition 3.12. (1) *For $i,j\in\{1,2\}$, the fusion rules of types*

$$\left(\begin{smallmatrix} & V_L^{T_j} & \\ V_L^{\pm} & & V_L^{T_i,\pm}\end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} & V_L^{T_j} & \\ V_L^{\pm} & & V_L^{T_i,\mp}\end{smallmatrix}\right)$$

are zero if $i\neq j$.
(2) *For $1\le r\le k-1$ and $i,j\in\{1,2\}$, the fusion rules of types*

$$\left(\begin{smallmatrix} & V_L^{T_j} & \\ V_{\lambda_r+L} & & V_L^{T_i,\pm}\end{smallmatrix}\right)$$

are zero if $(-1)^{r+\delta_{i,j}+1}\neq 1$.
(3) *For $i,j\in\{1,2\}$, the fusion rules of types*

$$\left(\begin{smallmatrix} & V_L^{T_j} & \\ V^{\pm}_{\alpha/2+L} & & V_L^{T_i,\pm}\end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} & V_L^{T_j} & \\ V^{\pm}_{\alpha/2+L} & & V_L^{T_i,\mp}\end{smallmatrix}\right)$$

is zero if $(-1)^{k+\delta_{i,j}+1}\neq 1$.

To prove Proposition 3.12, we first show the following proposition.

Proposition 3.13. *Let W be an irreducible V_L^+-module and suppose that W contains an $M(1)^+$-submodule isomorphic to $M(1,\lambda)$ for some $\lambda\in L^\circ$. If $(\lambda,i,j)\in\mathcal{P}_L$ is not a quasi-admissible triple, then fusion rules of types* $\left(\begin{smallmatrix} & V_L^{T_j} & \\ W & & V_L^{T_i,\pm}\end{smallmatrix}\right)$ *are zero.*

Proof. Let W be an irreducible V_L^+-module, and suppose that W contains an $M(1)^+$-submodule N isomorphic to $M(1,\lambda)$. Let f be an $M(1)^+$-isomorphism from $M(1,\lambda)$ to N. Consider an intertwining operator $\mathcal{Y} \in I_{V_L^+}\left(\begin{smallmatrix} & V_L^{T_j} & \\ W & & V_L^{T_i,\epsilon}\end{smallmatrix}\right)$ for $i,j \in \{1,2\}$ and $\epsilon \in \{\pm\}$. We shall prove that $\mathcal{Y} = 0$ if $(-1)^{\langle\alpha,\lambda\rangle+\delta_{i,j}+1} \neq 1$.

The restrictions of $\mathcal{Y}$ to $N \otimes V_L^{T_i,\epsilon}$ gives an intertwining operator for $M(1)^+$ of type $\left(\begin{smallmatrix} & V_L^{T_j} & \\ N & & V_L^{T_i,\epsilon}\end{smallmatrix}\right)$. Set

$$\overline{\mathcal{Y}}(u,z) = \phi_j^{-1}\mathcal{Y}(f(u),z)\phi_i \text{ for } u \in M(1,\lambda).$$

Then $\overline{\mathcal{Y}}$ is an intertwining operator for $M(1)^+$ of type $\left(\begin{smallmatrix} & M(1)(\theta) & \\ M(1,\lambda) & & M(1)(\theta)^\epsilon\end{smallmatrix}\right)$. By Lemma 3.8 (2), for any $u \in M(1,\lambda)$, $\overline{\mathcal{Y}}(u,z)$ is a linear combination of $\mathcal{Y}^\theta(u,z)$ and $\mathcal{Y}^\theta(\theta(u),z)$. By (3.14), we have

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{Y}^\theta(E,z_1)\mathcal{Y}^\theta(e_{\pm\lambda},z_2) - (-1)^{\langle\alpha,\lambda\rangle}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\mathcal{Y}^\theta(e_{\pm\lambda},z_2)\mathcal{Y}^\theta(E,z_1) \\ &\quad = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}^\theta(Y(E,z_0)e_{\pm\lambda},z_2), \end{aligned} \tag{3.17}$$

where $E = e_\alpha + e_{-\alpha} \in V_L^+$. Hence one have

$$(z_1-z_2)^M\mathcal{Y}^\theta(E,z_1)\mathcal{Y}^\theta(e_{\pm\lambda},z_2) = (-1)^{\langle\alpha,\lambda\rangle}(z_1-z_2)^M\mathcal{Y}^\theta(e_{\pm\lambda},z_2)\mathcal{Y}^\theta(E,z_1)$$

for a sufficiently large integer M, and then

$$(z_1-z_2)^M\mathcal{Y}^\theta(E,z_1)\overline{\mathcal{Y}}(e_\lambda,z_2) = (-1)^{\langle\alpha,\lambda\rangle}(z_1-z_2)^M\overline{\mathcal{Y}}(e_\lambda,z_2)\mathcal{Y}^\theta(E,z_1). \tag{3.18}$$

(3.18) is an identity on $M(1)(\theta)^\epsilon$. We next derive an identity on $V_L^{T_i,\epsilon}$ from (3.18). Since $e_{\pm\alpha} \in \mathbb{C}[L]$ act on $V_L^{T_i}$ $(i=1,2)$ as the scalar $(-1)^{\delta_{i,2}}$, we have

$$e_{\pm\alpha}\phi_j\overline{\mathcal{Y}}(u,z)\phi_i^{-1} = (-1)^{\delta_{i,j}+1}\phi_j\overline{\mathcal{Y}}(u,z)\phi_i^{-1}e_{\pm\alpha}$$

for $u \in M(1,\lambda)$. And $Y^\theta(E,z)$ acts on $V_L^{T_i}$ $(i=1,2)$ as $\mathcal{Y}^\theta(E,z)\otimes e_\alpha$. Hence by (3.18), we have

$$\begin{aligned} &(z_1-z_2)^M Y^\theta(E,z_1)\mathcal{Y}(f(e_\lambda),z_2) \\ &\quad = (-1)^{\langle\alpha,\lambda\rangle+\delta_{i,j}+1}(z_1-z_2)^M\mathcal{Y}(f(e_\lambda),z_2)Y^\theta(E,z_1) \end{aligned} \tag{3.19}$$

for a sufficiently large integer M. On the other hand, since $\mathcal{Y}$ is an intertwining operator for V_L^+ of type $\left(\begin{smallmatrix} & V_L^{T_j} & \\ W & & V_L^{T_i,\epsilon}\end{smallmatrix}\right)$, Jacobi identity (2.1) shows that

$$(z_1-z_2)^M Y^\theta(E,z_1)\mathcal{Y}(f(e_\lambda),z_2) = (z_1-z_2)^M \mathcal{Y}(f(e_\lambda),z_2)Y^\theta(E,z_1)$$

for a sufficiently large integer M. Therefore by (3.19) and (3.20), if $(-1)^{\langle\alpha,\lambda\rangle+\delta_{i,j}+1} \neq 1$, then

$$(z_1-z_2)^M \mathcal{Y}(f(e_\lambda),z_2)Y^\theta(E,z_1)u = 0 \tag{3.20}$$

for a nonzero $u \in V_L^{T_i,\epsilon}$ and a sufficiently large integer M. Since there is an integer n_0 such that $E_{n_0}u \neq 0$ and $E_n u = 0$ for all $n > n_0$, by multiplying $z_1^{n_0}$ and taking Res_{z_1} on both side of (3.20), we have $z_2^M \mathcal{Y}(f(e_\lambda),z_2)E_{n_0}u = 0$. Hence Lemma 2.3 implies that $\mathcal{Y} = 0$. $\square$

Now we prove Proposition 3.12.

Proof of Proposition 3.12. By the irreducible decompositions (3.3)-(3.5), we see that V_{λ_r+L} contains $M(1,\lambda_r)$ for $1 \leq r \leq k-1$, that $V_{\alpha/2+L}^{\pm}$ contain an $M(1)^+$-submodule isomorphic to $M(1,\alpha/2)$, and that $V_L^{\pm}$ contain an $M(1)^+$-submodule isomorphic to $M(1,\alpha)$. Hence Proposition 3.12 follows from Proposition 3.13. $\square$

Proposition 3.14. (1) *For $i \in \{1,2\}$, fusion rules of types*

$$\left(\begin{smallmatrix} & V_L^{T_i,\mp} & \\ V_L^+ & & V_L^{T_i,\pm}\end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} & V_L^{T_i,\pm} & \\ V_L^- & & V_L^{T_i,\pm}\end{smallmatrix}\right)$$

are zero.

(2) *Fusion rules of types*

$$\left(\begin{smallmatrix} & V_L^{T_1,\mp} & \\ V_{\alpha/2+L}^+ & & (V_L^{T_1,\pm})'\end{smallmatrix}\right), \left(\begin{smallmatrix} & V_L^{T_2,\pm} & \\ V_{\alpha/2+L}^+ & & (V_L^{T_2,\pm})'\end{smallmatrix}\right), \left(\begin{smallmatrix} & V_L^{T_1,\pm} & \\ V_{\alpha/2+L}^- & & (V_L^{T_1,\pm})'\end{smallmatrix}\right), \left(\begin{smallmatrix} & V_L^{T_2,\mp} & \\ V_{\alpha/2+L}^- & & (V_L^{T_2,\pm})'\end{smallmatrix}\right)$$

are zero.

Proof. Since $V_L^{\pm}$ contains the irreducible $M(1)^+$-module $M(1)^{\pm}$ respectively and the fusion rules of types $\left(\begin{smallmatrix} & M(1)(\theta)^{\mp} & \\ M(1)^+ & & M(1)(\theta)^{\pm}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} & M(1)(\theta)^{\pm} & \\ M(1)^- & & M(1)(\theta)^{\pm}\end{smallmatrix}\right)$ are zero by Theorem 2.7 (i) and (ii), (1) follows from Corollary 2.4.

Next we prove that the fusion rules of types $\left(\begin{smallmatrix} & V_L^{T_1,\mp} & \\ V_{\alpha/2+L}^+ & & (V_L^{T_1,\pm})'\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} & V_L^{T_2,\pm} & \\ V_{\alpha/2+L}^+ & & (V_L^{T_2,\pm})'\end{smallmatrix}\right)$ are zero. (2) for types $\left(\begin{smallmatrix} & V_L^{T_1,\pm} & \\ V_{\alpha/2+L}^- & & (V_L^{T_1,\pm})'\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} & V_L^{T_2,\mp} & \\ V_{\alpha/2+L}^- & & (V_L^{T_2,\pm})'\end{smallmatrix}\right)$ can be also proved in the similar way.

By Proposition 3.11 (iii), for $i \in \{1,2\}$ and $\epsilon \in \{\pm\}$, there exists $\epsilon' \in \{\pm\}$ such that the fusion rule of type $\binom{V_L^{T_i,\epsilon'}}{V_{\alpha/2+L}^+ \ (V_L^{T_i,\epsilon})'}$ is nonzero. Let $\{\tau, \epsilon'\} = \{\pm\}$. Then we have to prove that the fusion rule of type $\binom{V_L^{T_i,\tau}}{V_{\alpha/2+L}^+ \ (V_L^{T_i,\epsilon})'}$ is zero. To show this, we prove that the canonical projection

$$\binom{V_L^{T_i}}{V_{\alpha/2+L}^+ \ (V_L^{T_i,\epsilon})'} \to \binom{V_L^{T_i,\epsilon'}}{V_{\alpha/2+L}^+ \ (V_L^{T_i,\epsilon})'},\ \mathcal{Y} \mapsto p_{\epsilon'} \circ \mathcal{Y}$$

is injective, where $p_\pm$ are the canonical projection from $V_L^{T_i}$ to $V_L^{T_i,\pm}$ respectively and $p_\pm \circ \mathcal{Y}$ are intertwining operators defined by $(p_\pm \circ \mathcal{Y})(u,z)v = p_\pm(\mathcal{Y}(u,z)v)$ for $u \in V_{\alpha/2+L}^+$ and $v \in (V_L^{T_i,\epsilon})'$. To prove this, it is enough to prove that arbitrary nonzero intertwining operator $\mathcal{Y}$ of type $\binom{V_L^{T_i}}{V_{\alpha/2+L}^+ \ (V_L^{T_i,\epsilon})'}$ satisfies

$$\theta\mathcal{Y}(e_{\alpha/2} + e_{-\alpha/2}, z)\theta = (-1)^{\delta_{i,2}}\mathcal{Y}(e_{\alpha/2} + e_{-\alpha/2}, z). \tag{3.21}$$

Actually if $\mathcal{Y}$ is a nonzero intertwining operator of the indicated type which satisfies (3.21), then $p_{\epsilon'}(\mathcal{Y}(e_{\alpha/2} + e_{-\alpha/2}, z)v) = \mathcal{Y}(e_{\alpha/2} + e_{-\alpha/2}, z)v$ for $v \in (V_L^{T_i,\epsilon})'$ and then $p_{\epsilon'} \circ \mathcal{Y}$ is nonzero by Corollary 2.4.

Let $V_L^{T_j} = (V_L^{T_i})'$, and let $V_{\alpha/2+L}^+[0]$ be as of (3.8). Then $\mathcal{Y}$ gives an intertwining operator for $M(1)^+$ of type $\binom{V_L^{T_i}}{V_{\alpha/2+L}^+[0] \ \ V_L^{T_j,\epsilon}}$. Thus by Lemma 3.8 (2), we see that $I_{M(1)^+}\binom{V_L^{T_j}}{V_{\alpha/2+L}^+[0] \ \ V_L^{T_i,\epsilon}}$ is spanned by intertwining operators $\mathcal{Y}^\pm$ defined by

$$\mathcal{Y}^\pm(u,z) = \phi_i\mathcal{Y}^\theta(\phi_{\pm\alpha/2}(u), z)\phi_j^{-1} \text{ for } u \in V_{\alpha/2+L}^+[0]. \tag{3.22}$$

Hence there exist constants $c_1, c_2 \in \mathbb{C}$ such that

$$\mathcal{Y}(u,z) = c_1\mathcal{Y}^+(u,z) + c_2\mathcal{Y}^-(u,z) \tag{3.23}$$

for all $u \in V_{\alpha/2+L}^+[0]$. Now for $\beta \in \mathfrak{h}$, set

$$\exp\left(\sum_{n=0}^{\infty} \frac{\beta(-n)}{n} z^n\right) = \sum_{n=0}^{\infty} p_n(\beta) z^n \in (\operatorname{End} V_{L^\circ})[[z]].$$

Then we have $E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}) = p_{k-1}(\alpha)e_{\frac{\alpha}{2}} + p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}} \in V_{\alpha/2+L}^+[0]$, and hence

$$\phi_{\frac{\alpha}{2}}(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}})) = p_{k-1}(\alpha)e_{\frac{\alpha}{2}}, \quad \phi_{-\frac{\alpha}{2}}(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}})) = p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}}.$$

Thus by (3.22) and (3.23), we have

$$\begin{aligned}
&[E_0, \mathcal{Y}(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}, z)] \\
&\quad = \mathcal{Y}(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), z) \\
&\quad = \phi_i(c_1\mathcal{Y}^\theta(p_{k-1}(\alpha)e_{\frac{\alpha}{2}}, z) + c_2\mathcal{Y}^\theta(p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}}, z))\phi_j^{-1}. \qquad (3.24)
\end{aligned}$$

On the other hand, (3.17) shows that

$$\begin{aligned}
[E_0, \phi_i\mathcal{Y}^\theta(e_{\pm\frac{\alpha}{2}}, z)\phi_j^{-1}] &= e_\alpha\phi_i\mathcal{Y}^\theta(E_0(e_{\pm\frac{\alpha}{2}}), z)\phi_j^{-1} \\
&= (-1)^{\delta_{i,2}}\phi_i\mathcal{Y}^\theta(p_{k-1}(\mp\alpha)e_{\mp\frac{\alpha}{2}}, z)\phi_j^{-1}.
\end{aligned}$$

Hence by (3.22) and (3.23) again, we have

$$\begin{aligned}
&[E_0, \mathcal{Y}(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}, z)] \\
&\quad = (-1)^{\delta_{i,2}}c_1([E_0, \mathcal{Y}^+(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), z)] + c_2[E_0, \mathcal{Y}^-(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), z)]) \\
&\quad = (-1)^{\delta_{i,2}}\phi_i(c_1\mathcal{Y}^\theta(p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}}, z) + c_2\mathcal{Y}^\theta(p_{k-1}(\alpha)e_{\frac{\alpha}{2}}, z))\phi_j^{-1}. \qquad (3.25)
\end{aligned}$$

Subtracting (3.24) from (3.25) gives the identity

$$(c_1 - (-1)^{\delta_{i,2}}c_2)\phi_i(\mathcal{Y}^\theta(p_{k-1}(\alpha)e_{\frac{\alpha}{2}}, z) - (-1)^{\delta_{i,2}}\mathcal{Y}^\theta(p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}}, z))\phi_j^{-1} = 0.$$

Then Lemma 3.8 shows that $c_1 = (-1)^{\delta_{i,2}}c_2$. Since $\theta\mathcal{Y}^\pm(u, z)\theta = \mathcal{Y}^\mp(u, z)$ for $u \in V_{\alpha/2+L}^+[0]$, we have (3.21). $\square$

Now the following proposition follows from Proposition 2.2, Proposition 3.3, Proposition 3.11, Proposition 3.12 and Proposition 3.14.

Proposition 3.15. *Let W^1, W^2 and W^3 be irreducible V_L^+-modules and suppose that one of them is of twisted type. Then the fusion rule $N_{W^1W^2}^{W^3}$ is zero or one. Assume that W^1 is twisted type module, then the fusion rule $N_{W^1W^2}^{W^3}$ is one if and only if W^i $(i = 1, 2, 3)$ satisfy the following cases*:
(i) $W^1 = (V_L^{T_1,+})'$ *and the pair (W^2, W^3) is one of pairs*

$$\begin{aligned}
&(V_L^\pm, (V_L^{T_1,\pm})'),\ (V_L^{T_1,\pm}, V_L^\pm),\ (V_{\alpha/2+L}^\pm, V_L^{T_1,\pm}),\ ((V_L^{T_1,\pm})', (V_{\alpha/2+L}^\pm)'), \\
&(V_{\lambda_r+L}, (V_L^{T_1,\pm})') \text{ and } (V_L^{T_1,\pm}, V_{\lambda_r+L}) \text{ for } 1 \le r \le k-1 \text{ and } r \text{ is even}, \\
&(V_{\lambda_r+L}, (V_L^{T_2,\pm})') \text{ and } (V_L^{T_2,\pm}, V_{\lambda_r+L}) \text{ for } 1 \le r \le k-1 \text{ and } r \text{ is odd.}
\end{aligned}$$

(ii) $W^1 = (V_L^{T_1,-})'$ *and the pair* (W^2, W^3) *is one of pairs*

$$(V_L^{\pm}, (V_L^{T_1,\mp})'),\ (V_L^{T_1,\pm}, V_L^{\mp}),\ (V_{\alpha/2+L}^{\pm}, V_L^{T_1,\mp}),\ ((V_L^{T_1,\pm})', (V_{\alpha/2+L}^{\mp})'),$$
$$(V_{\lambda_r+L}, (V_L^{T_1,\pm})') \text{ and } (V_L^{T_1,\pm}, V_{\lambda_r+L}) \text{ for } 1 \le r \le k-1 \text{ and } r \text{ is even},$$
$$(V_{\lambda_r+L}, (V_L^{T_2,\pm})') \text{ and } (V_L^{T_2,\pm}, V_{\lambda_r+L}) \text{ for } 1 \le r \le k-1 \text{ and } r \text{ is odd.}$$

(iii) $W^1 = (V_L^{T_2,+})'$ *and the pair* (W^2, W^3) *is one of pairs*

$$(V_L^{\pm}, (V_L^{T_2,\pm})'),\ (V_L^{T_2,\pm}, V_L^{\pm}),\ (V_{\alpha/2+L}^{\pm}, V_L^{T_2,\mp}),\ ((V_L^{T_2,\pm})', (V_{\alpha/2+L}^{\mp})'),$$
$$(V_{\lambda_r+L}, (V_L^{T_2,\pm})') \text{ and } (V_L^{T_2,\pm}, V_{\lambda_r+L}) \text{ for } 1 \le r \le k-1 \text{ and } r \text{ is even},$$
$$(V_{\lambda_r+L}, (V_L^{T_1,\pm})') \text{ and } (V_L^{T_1,\pm}, V_{\lambda_r+L}) \text{ for } 1 \le r \le k-1 \text{ and } r \text{ is odd.}$$

(iv) $W^1 = (V_L^{T_2,-})'$ *and the pair* (W^2, W^3) *is one of pairs*

$$(V_L^{\pm}, (V_L^{T_2,\mp})'),\ (V_L^{T_2,\pm}, V_L^{\mp}),\ (V_{\alpha/2+L}^{\pm}, V_L^{T_2,\pm}),\ ((V_L^{T_2,\pm})', (V_{\alpha/2+L}^{\pm})'),$$
$$(V_{\lambda_r+L}, (V_L^{T_2,\pm})') \text{ and } (V_L^{T_2,\pm}, V_{\lambda_r+L}) \text{ for } 1 \le r \le k-1 \text{ and } r \text{ is even},$$
$$(V_{\lambda_r+L}, (V_L^{T_1,\pm})') \text{ and } (V_L^{T_1,\pm}, V_{\lambda_r+L}) \text{ for } 1 \le r \le k-1 \text{ and } r \text{ is odd.}$$

Consequently Theorem 3.4 follows from Proposition 2.2, Proposition 3.7 and Proposition 3.15.

ACKNOWLEDGMENT

I would like to thank Professor Kiyokazu Nagatomo for helpful advice.

References

[A] T. Abe, Fusion rules for the free bosonic orbifold vertex operator algebra, J. Algebra, to appear.

[D1] C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **160** (1993), 245-265.

[D2] C. Dong, Twisted modules for vertex algebras associated with even lattices, *J. Algebra* **165** (1994), 90-112.

[DG] C. Dong and R. L. Griess, Rank one lattice type vertex operator algebras and their automorphism groups, *J. Algebra* **208** (1998), 262-275.

[DL] C. Dong and J. Lepowsky, "Generalized vertex algebras and relative vertex operators", Progress in Math., Vol.112, Birkhäuser, Boston, 1993.

[DLM] C. Dong, H.-S. Li and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571-600.

[DM] C. Dong and G. Mason, On quantum Galois theory, *Duke Math. J.* **86**, No. 2 (1997), 305-321.

[DMZ] C. Dong, G. Mason and Y.-C. Zhu, Discrete series of the Virasoro algebra and the moonshine module, in "Algebraic Groups and Their Generalizations: Quantum and Infinite-Dimensional Methods" (W. J. Haboush, B. J. Parshall, Eds.) Vol. **56**, Pt. 2, pp. 295-316, Amer. Math. Soc., Providence, RI, 1994.

[DN1] C. Dong and K. Nagatomo, Classification of irreducible modules for the vertex operator algebra $M(1)^+$, *J. Algebra* **216** (1999), 384-404.

[DN2] C. Dong and K. Nagatomo, Representations of Vertex operator algebra V_L^+ for rank one lattice L, *Commun. Math. Phys.* **202** (1999), 169-195.

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Amer. Math. Soc. **104** (1993).

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, "Vertex Operator Algebras and the Monster", Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[FZ] I. Frenkel and Y.-C. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123-168.

[HL] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra I,II, *Selecta Math., New Series* **1** (1995), 757-786.

[KT] V. Kac and I. Todorov, Affine orbifolds and rational conformal field theory extensions of $W_{1+\infty}$, *Commun. Math. Phys.* **190** (1997), 57-111.

[Z] Y.-C. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237-302.